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Rearrangement and Subseries Numbers

Master's Thesis

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Abstract

By rearranging the terms of a conditionally convergent series we can make it assume a different limit or even diverge. Similarly we could do so by taking a subseries of a conditionally convergent series. The recently studied *rearrangement* and *subseries* numbers are the least number of permutations or subsets of indices that are needed to change the behaviour of every conditionally convergent series. The rearrangement and subseries numbers are cardinal characteristics (uncountable cardinalities bound from above by the cardinality of the continuum). In this thesis we will investigate their dual cardinal characteristics, that is, the least number of conditionally convergent series needed such that no single permutation or subset of indices alters the behaviour of all of the series simultaneously. We show that most of the results known about the rearrangement and subseries numbers correspond naturally to dual statements about their dual cardinal characteristics. Additionally we formulate the subseries numbers in an alternative way that gives rise to some new subseries numbers, and prove a few original results about them.

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Introduction

Cardinal characteristics of the continuum are cardinalities that lie in between $\aleph_1$ and the continuum $\mathfrak{c} = 2^{\aleph_0}$. When we assume that that the continuum hypothesis fails, we have a strict inequality $\aleph_1 < \mathfrak{c}$. With the method of forcing, Paul Cohen [8, 9] showed that the failing of the continuum hypothesis is consistent with ZFC. This opened the way for studying which cardinal characteristics could be consistently strictly larger than other cardinal characteristics.

In this thesis we will take a look at two families of cardinal characteristics that were formulated by Andreas Blass, Jörg Brendle, Will Brian, Joel David Hamkins, Michael Hardy and Paul B. Larson in the rearrangement [4] and subseries [7] papers. Both families of cardinal characteristics are formulated by studying how the convergence or divergence of infinite series of real numbers can be influenced, either by rearranging the terms of the series or by taking a subseries.

A conditionally convergent series is an infinite series $\sum_n a_n$ that converges, but for which the absolute series $\sum_n |a_n|$ diverges. Bernhard Riemann showed with his rearrangement theorem [30] that for any conditionally convergent series $\sum_n a_n$ there exists a permutation on the natural numbers π such that $\sum_n a_{\pi(n)}$ no longer converges to the same limit. Indeed, it is possible to rearrange the terms of any conditionally convergent series, and make it converge to any new limit, make it diverge to positive or negative infinity or to make it diverge by oscillation. A similar statement is true for taking subseries of a conditionally convergent series. Inspired by Riemann's rearrangement theorem, we define the **rearrangement number** and the **subseries number**:

Definition — *Rearrangement number*
The *rearrangement number* $\mathfrak{rr}$ is the smallest cardinality of a family $\Pi \subseteq \mathcal{S}(\omega)$ of permutations on ω such that for any conditionally convergent series $\sum_n a_n$ there is a permutation $\pi \in \Pi$ such that $\sum_n a_{\pi(n)}$ does not converge to the same limit. $\triangleleft$

Definition — *Subseries number*
The *subseries number* $\mathfrak{ß}$ is the smallest cardinality of a family $\mathcal{A} \subseteq [\omega]^\omega$ of infinite subsets of ω such that for any conditionally convergent series $\sum_{n\in\omega} a_n$ there is an $A \in \mathcal{A}$ for which $\sum_{n\in A} a_n$ diverges. $\triangleleft$

We can be more specific and define (i) $\mathfrak{rr}_f$, (ii) $\mathfrak{rr}_i$ and (iii) $\mathfrak{rr}_o$ as the smallest cardinality of a family of permutations such that every conditionally convergent series (i) converges to a different finite sum, (ii) diverges to infinity, or (iii) diverges by oscillation. We can also define $\mathfrak{rr}_{fi}$, $\mathfrak{rr}_{fo}$ and $\mathfrak{rr}_{io}$ for the combinations of these conditions. Similarly we can define (i) $\mathfrak{ß}_i$ and (ii) $\mathfrak{ß}_o$ as the smallest cardinality of a family of subsets of ω such that every conditionally convergent series (i) diverges to infinity, or (ii) diverges by oscillation.

By a diagonalisation argument it is not difficult to see that all rearrangement and subseries numbers are uncountable, and since the set $\mathcal{S}(\omega)$ of permutations on ω and the set $[\omega]^\omega$ of infinite subsets of ω are both of cardinality $2^{\aleph_0}$, we see that all rearrangement and subseries numbers are bound from above by the continuum. Furthermore, in [4] it is shown that $\mathfrak{rr} = \mathfrak{rr}_o = \mathfrak{rr}_{io} = \mathfrak{rr}_{fo}$, which reduces the number of different rearrangement numbers significantly.

Some cardinal characteristics, including all the cardinal characteristics we have seen above, can be expressed using **relational systems**. These relational systems consist of two sets A and B, and a relation $R \subseteq A \times B$. We can define a cardinality by considering the least cardinality of a subset $B' \subseteq B$ such that every $a \in A$ has some $b \in B'$ for which $a \mathrel{R} b$. This cardinality is called the **norm** of the relational system.

There is a natural way to define a **dual** cardinal characteristic by considering the least cardinality of a subset $A' \subseteq A$ such that there exists no $b \in B$ for which all $a \in A'$ have $a \mathrel{R} b$. Since we can formulate the rearrangement and subseries numbers as the norms of relational systems, we can define the dual rearrangement and dual subseries numbers:

Definition — *Dual rearrangement number*
The *dual rearrangement number* $\mathfrak{rr}^\perp$ is the smallest set C of conditionally convergent series such that every $\pi \in \mathcal{S}(\omega)$ has a series $\sum_n a_n \in C$ such that $\sum_n a_n = \sum_n a_{\pi(n)}$. ◁

Definition — *Dual subseries number*
The *dual subseries number* $\mathfrak{ß}^\perp$ is the smallest set C of conditionally convergent series s such that every infinite $S \in [\omega]^\omega$ has a series $\sum_{n\in\omega} a_n \in C$ such that $\sum_{n\in S} a_n$ converges. ◁

Once again we can define $\mathfrak{rr}_i^\perp$ as the smallest set of conditionally convergent series for which every permutation has a series that does not diverge to infinity, and similar for the other numbers.

The main focus of this thesis will be towards investigating the properties of the dual rearrangement and dual subseries numbers. Many of the results from the two papers can be described in terms of relational systems. By formulating a specific kind of morphism between relational systems we could prove statements about the ordering of their norms for both the original relational systems and the dual relational systems. By exploiting this, we will see that many of the results from the papers are provable in their dual form.

Not all the proofs from the two papers can be translated. We will discover that the proof for $\mathfrak{rr}_o = \mathfrak{rr}$ is not directly translatable. However, we will make use of another cardinal characteristic to give an alternative proof that $\mathfrak{rr}_o = \mathfrak{rr}$, and this new proof will dualise.

The rearrangement paper also contains a proof of the consistency of $\mathfrak{rr}_i < \mathfrak{c}$ and $\mathfrak{rr}_f < \mathfrak{c}$ using the method of iterated forcing. In this thesis we will see that the dual statements $\aleph_1 < \mathfrak{rr}_i^\perp$ and $\aleph_1 < \mathfrak{rr}_f^\perp$ are provable as well by using a different iteration of the same forcings from the papers.

Finally, in the subseries paper it is argued that there is no natural analogue $\mathfrak{ß}_f$ of the rearrangement number $\mathfrak{rr}_f$. The rationale behind this thought is that getting a subseries to diverge to a different finite sum is very easy by omitting one of the nonzero terms of the series, while such a finite change to a permutation will have no effect. In this thesis we will slightly adjust the definition of the subseries numbers by excluding all cofinite subsets of ω from the family $\mathcal{A}$. We will show that this does not affect the size of $\mathfrak{ß}$, $\mathfrak{ß}_i$ or $\mathfrak{ß}_o$, and moreover that this gives us a way to look at $\mathfrak{ß}_f$ more sensibly.

The first chapter of this thesis will give an overview of the set theoretic tools that will be used, in particular it will treat the basics of descriptive set theory and of Cohen's method of forcing. The second chapter is meant as an introduction to cardinal characteristics and relational systems and furthermore gives an overview of six models of ZFC that will help us in proving the consistency of strict inequalities between cardinal characteristics. The third chapter is devoted to the rearrangement numbers and the fourth chapter to the subseries numbers.

For easy reference an overview of relational systems is included at the end, as well as an index of symbols and an index of terms.

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CHAPTER 1

PRELIMINARIES

This chapter will contain a brief summary of preliminary concepts and results in Zermelo-Fraenkel set theory. It is assumed the reader is already familiar with the basics of axiomatic set theory. Knowledge of descriptive set theory and of forcing is useful but not essential. We will introduce terminology of basic set theoretic notions in the first section. The second section will treat some basic notions from descriptive set theory and the third section will introduce the method of forcing.

1.1 TERMINOLOGY

Throughout this thesis we mostly work with **Zermelo-Fraenkel Set Theory** (abbreviated as ZFC), that is, the first-order theory with a single relation symbol $\in$ consisting of the axioms of Zermelo-Fraenkel including the Axiom of Choice (which we abbreviate as AC). We will use ZF to denote the theory consisting of the axioms of Zermelo-Fraenkel excluding AC.

We abbreviate $\forall x(x \in y \to \psi)$ as $\forall x \in y(\psi)$ and $\exists x(x \in y \wedge \psi)$ as $\exists x \in y(\psi)$.

The **empty set** is denoted as $\varnothing$. We write $x \subseteq y$ if x is a subset of y and $x \subsetneq y$ if the subset is proper. Furthermore, $x \cup y$, $x \cap y$, $x \setminus y$ and $x \triangle y$ denote the **union**, **intersection**, **relative complement** and **symmetric difference** respectively.

Ordinals are denoted by the lowercase Greek alphabet. Arbitrary ordinals are usually meant with the letters $\alpha, \beta, \gamma, \delta$ or ξ, while **cardinals** are usually meant with the letters κ, λ, μ. The aleph numbers $\aleph_\alpha$ are written as ω_α when we use them as ordinals instead of cardinals. The class of ordinals is denoted by Ord. We denote the **cofinality** of an ordinal α as $\mathrm{cf}(\alpha)$. **Regular** cardinals are those cardinals κ with $\mathrm{cf}(\kappa) = \kappa$.

The **power set** of a set x is denoted as $\mathcal{P}(x)$. The set of subsets in $\mathcal{P}(x)$ of a certain cardinality κ are written as $[x]^\kappa = \{y \in \mathcal{P}(x) \mid |y| = \kappa\}$, and the set of subsets strictly smaller than κ are written as $[x]^{<\kappa}$. The **cartesian product** of sets x and y is written as $x \times y$. In case we need to take an infinite cartesian product over some family $\{X_i \mid i \in I\}$, we write $\prod_{i\in I} X_i$. If all $X_i = X$ are equal, we write ${}^I X$ to denote the cartesian product. We could regard ${}^I X$ as the **set of functions** from I to X or as the set of **sequences** of elements in X indexed by I, and we will indeed consider products, functions and sequences as being practically the same thing.

If R is a relation, we write $R^{-1} = \{(a, b) \mid (b, a) \in R\}$ for the **inverse relation**. We also write $(x, y) \in R$ in infix notation as $x\ R\ y$. The complement of a relation $R \subseteq A \times B$ is written as $R^c = (A \times B) \setminus R$.

If $f : X \to Y$ is a function, we write $(x, y) \in f$ alternatively as $f(x) = y$ or as $f : x \mapsto y$. When $a \subseteq X$ we write $f[a] = \{f(x) \mid x \in a\}$ for the **image** of f over a, and we will write $\mathrm{dom}(f) = X$ and $\mathrm{ran}(f) = f[X]$ respectively for the **domain** and **range** of f. Like with relations, for injective functions we will also write f^{-1} to denote the **inverse** function. The **composition** of functions $f : X \to Y$ and $g : Y \to Z$ is written as $g \circ f : X \to Z$, given by $f \circ g : x \mapsto g(f(x))$. If $a \subseteq X$ we write $f \restriction a = \{(x, y) \in f \mid x \in a\}$ for the **restriction** of the domain of f to a.

If $f : X \to Y$ is a function, and $g \subseteq f$, we say g is a **partial function** from X to Y, written as $g : X \nrightarrow Y$. The set of partial function from X to Y with a domain of cardinality less that κ is written as $\mathrm{Fn}_\kappa(X, Y)$, for example $\mathrm{Fn}_{\aleph_0}(\omega, \omega)$ is the set of finite partial functions from ω to itself. If α is an ordinal, then ${}^\alpha x$ is the set of functions from α to x, also regarded as a **sequence** of elements in x of length α. The set ${}^{<\alpha}x$ is shorthand for $\bigcup_{\beta<\alpha} {}^\beta x$, that is, the set of initial segments of sequences in ${}^\alpha x$. If α and β are ordered sets, we denote the set of **strictly increasing** functions from α to β as $\uparrow({}^\alpha\beta)$.

A **countable** set is a set with a cardinality smaller than or equal to $\aleph_0$. If x is infinite, then a subset $a \subseteq x$ is **cofinite** if $x \setminus a$ is finite, and **coinfinite** if $x \setminus a$ is infinite. If a property φ holds for **almost all** elements in a set x, we mean that $\{a \in x \mid \varphi(x)\}$ is cofinite, and we write this with the shorthand $\forall^\infty x \in X(\varphi(x))$. Dually we have $\exists^\infty x \in X(\varphi(x))$, with the meaning that φ holds for an infinite number of elements of X.

The notation $A =^* B$ is used to say that A is almost equal to B, which means that the symmetric difference $A \triangle B$ is finite. In a similar fashion we define $A \subseteq^* B$ to mean that A is a subset of B except for finitely many elements, that is, $A \setminus B$ is finite.

1.2 Descriptive Set Theory

As we are interested in properties of the real continuum $\mathbb{R}$, we will spend this section to describe some properties of $\mathbb{R}$. However, for several purposes it is often more useful to work with spaces that are closely related to $\mathbb{R}$, such as $[0, 1]$, $(0, 1)$, ${}^\omega 2$, ${}^\omega\omega$, $\mathcal{P}(\omega)$ and $[\omega]^\omega$. All of these sets have the same cardinality as $\mathbb{R}$, and admit topologies such that each of them is almost homeomorphic to any other (that is, they are homeomorphic after a countable subset is removed). Furthermore, many useful properties pertaining to later concepts are preserved under such almost homeomorphisms.

Definition 1.2.1 — *Polish spaces*
A **Polish space** is a topological space on which a complete metric can be defined that has a countable (topologically) dense subset. Note that the metric does not have to be explicitly fixed for a Polish space, there just needs to exist one. A space is **perfect** if no open set is a singleton. ◁

Definition 1.2.2 — *Integers & rational numbers*
Let $\sim_{\mathbb{Z}}$ be the relation on $\omega \times \omega$ such that $(n, m) \sim_{\mathbb{Z}} (n', m')$ if and only if $n + m' = n' + m$. The set of **integers** $\mathbb{Z}$ is defined as the quotient set $(\omega \times \omega)/{\sim_{\mathbb{Z}}}$.

Let $\sim_{\mathbb{Q}}$ be the relation on $\mathbb{Z} \times (\omega \setminus \{0\})$ such that $(a, b) \sim_{\mathbb{Q}} (a', b')$ if and only if $a \cdot b' = a' \cdot b$. The set of **rational numbers** $\mathbb{Q}$ is defined as the quotient set $(\mathbb{Z} \times (\omega \setminus \{0\}))/{\sim_{\mathbb{Q}}}$. ◁

We define an ordering on $\mathbb{Z}$ and $\mathbb{Q}$ as follows: $(n, m) < (n', m')$ in $\mathbb{Z}$ if and only if $n+m' < n'+m$, and $(a, b) < (a', b')$ in $\mathbb{Q}$ if and only if $a \cdot b' < a' \cdot b$ in $\mathbb{Z}$. Arithmetic is defined in the standard way, such that the maps $\omega \to \mathbb{Z} : n \mapsto (n, 0)$ and $\mathbb{Z} \to \mathbb{Q} : a \mapsto (a, (1, 0))$ are embeddings preserving order and arithmetic. We will identify the images of these maps with their preimage, and thus sometimes talk about $n \in \omega$ as if they are integers, and about $a \in \mathbb{Z}$ as if they are rational numbers, given the right context. Similarly we will sometimes talk about rational numbers as if they are real numbers.

Definition 1.2.3 — *Real numbers*
A **Dedekind cut** is a subset $S \subseteq \mathcal{P}(\mathbb{Q})$ such that $S \neq \varnothing$ and $S \neq \mathbb{Q}$, S is downward closed and S contains no maximal element. The set of **real numbers** $\mathbb{R}$ is defined as the set of all Dedekind cuts on $\mathbb{Q}$. If $r, r' \in \mathbb{R}$ are reals represented by the Dedekind cuts S, S' respectively, then $r \leq r'$ if and only if $S \subseteq S'$. A real number $r \in \mathbb{R}$ is **rational** if the Dedekind cut S represented by r has a supremum inside $\mathbb{Q}$, otherwise it is **irrational**.

The rules of arithmetic for $\mathbb{R}$ are defined as usual. The map $\mathbb{Q} \to \mathbb{R}$ sending $q \mapsto \{p \in \mathbb{Q} \mid p < q\}$ is an embedding that preserves order and arithmetic. As before, we will not make the distinction between rational numbers and rational real numbers.

The standard topology on $\mathbb{R}$ is given by the order topology, or equivalently by the metric topology using metric $d(x, y) = |x - y|$. ◁

When we talk about *reals*, we usually mean any element of $\mathbb{R}$. However, as it is often convenient to work with other spaces, we use the term *real* freely to denote any element of a space suitably similar to $\mathbb{R}$. We will give a few examples of such spaces. The most natural are the intervals $(0, 1)$ and $[0, 1]$ with the subspace topology inherited from $\mathbb{R}$. We will introduce some other spaces now.

Definition 1.2.4 — *Cantor space*
The **Cantor space** is the set ${}^{\omega}2$ of functions from ω to the set $2 = \{0, 1\}$. The standard topology on the Cantor space is given by product topology of ${}^{\omega}2$, where $\{0, 1\}$ has the discrete topology. For $s \in 2^{<\omega}$, define $U_s = \{f \in {}^{\omega}2 \mid \forall n \in \mathrm{dom}(s)(f(n) = s(n))\}$, then $\{U_s \mid s \in 2^{<\omega}\}$ is a basis of clopens for the topology. ◁

Definition 1.2.5 — *Baire space*

The **Baire space** is the set ${}^{\omega}\omega$ of functions from ω to ω. As with the Cantor space, the standard topology on the Baire space is the product topology of ${}^{\omega}\omega$, where ω has the discrete topology. For $s \in \omega^{<\omega}$, let $U_s = \{f \in {}^{\omega}\omega \mid \forall n \in \mathrm{dom}(s)(f(n) = s(n))\}$, then $\{U_s \mid s \in \omega^{<\omega}\}$ is a basis of clopens for the topology. This topology is equivalent to the metric topology given by $d(f,g) = \frac{1}{\min\{n \mid f(n)\neq g(n)\}+1}$ if $f \neq g$ and $d(f,g) = 0$ if $f = g$. $\triangleleft$

Proposition 1.2.6

Each of the spaces $\mathbb{R}$, ${}^{\omega}2$ and ${}^{\omega}\omega$ is a perfect Polish space. $\triangleleft$

Definition 1.2.7 — *G_δ and F_σ sets*

A $\mathbf{G}_\delta$ **set** is a countable intersection of open sets and an $\mathbf{F}_\sigma$ **set** is a countable union of closed sets. $\triangleleft$

Proposition 1.2.8

A subspace X of a Polish space Y is a Polish space if and only if X is a G_δ set in Y. $\triangleleft$

Proposition 1.2.9

The space $\mathcal{P}(\omega)$ is a perfect Polish space homeomorphic to ${}^{\omega}2$ by the map $\mathcal{P}(\omega) \to {}^{\omega}2$ mapping $A \mapsto \chi_A$, where χ_A is the characteristic map, i.e. $\chi_A(n) = 1$ if and only if $n \in A$.

Let $[\omega]^{\omega}$ be the **set of infinite subsets** of ω, let $[\omega]^{\omega}_{\omega}$ be the **set of infinite coinfinite subsets** of ω and let $\mathcal{S}(\omega)$ be the **set of permutations** on ω (with *permutation* we mean a bijection from ω to itself). Both $[\omega]^{\omega}$ and $[\omega]^{\omega}_{\omega}$ are perfect Polish spaces as G_δ subspaces of $\mathcal{P}(\omega)$ and $\mathcal{S}(\omega)$ is a perfect Polish space as a G_δ subspace of ${}^{\omega}\omega$. $\triangleleft$

Instead of working with ${}^{\omega}2$ we might as well work with ${}^{\omega}n$ for any $n \in \omega$, or indeed with an infinite product space of any sequence of natural numbers.

Proposition 1.2.10

If $\overline{n}, \overline{m} \in {}^{\omega}(\omega \setminus \{0,1\})$ are sequences of natural numbers greater than or equal to 2, and the product spaces $S = \prod_{k\in\omega} n_k$ and $T = \prod_{k\in\omega} m_k$ are given the product topology with each n_k and m_k having the discrete topology, then S and T are homeomorphic. $\triangleleft$

Definition 1.2.11 — *Cardinality of the continuum*

The **cardinality of the continuum** is defined as $\mathfrak{c} = 2^{\aleph_0}$. The **Continuum Hypothesis** states that $\mathfrak{c} = \aleph_1$, and is denoted as CH. The **Generalised Continuum Hypothesis**, GCH, states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α. $\triangleleft$

Theorem 1.2.12

Every nonempty perfect Polish space has cardinality $2^{\aleph_0}$. $\triangleleft$

Hence all the spaces we have seen so far have the same cardinality.

Definition 1.2.13
A (proper) **ideal** of the reals is a (proper) subset $I \subseteq \mathcal{P}(\mathbb{R})$ such that I is nonempty and closed under subsets. A **σ-ideal** is an ideal that is also closed under countable unions. $\triangleleft$

Ideals can be seen as a description of *smallness*. A subset $A \subseteq \mathbb{R}$ is small in the sense of an ideal I when $A \in I$. There are two σ-ideals of the reals that are of main importance to us, namely the ideal of Lebesgue null sets and the ideal of meagre sets, also known as sets of first category.

Definition 1.2.14 — *Measure*
A **σ-algebra** over a set X is a family $\Sigma \subseteq \mathcal{P}(X)$ such that $X \in \Sigma$ and Σ is closed under complements and countable unions. A **measure** $\mu : \Sigma \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a function such that $\mu(\varnothing) = 0$ and for any countable disjoint family $\{A_i \mid i \in \omega\} \subseteq \Sigma$ we have $\mu(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega}\mu(A_i)$. A **measure space** $\langle X, \Sigma, \mu\rangle$ is a set X with a σ-algebra Σ over X and a measure μ on Σ. A set $A \in \Sigma$ is a **μ-null set** if $\mu(A) = 0$.

A **Borel measure** on a space X with topology τ is a measure μ of a measure space $\langle X, \Sigma, \mu\rangle$, where Σ is the smallest σ-algebra that contains τ. A measure μ is a **complete measure** if the set of μ-null sets is closed under subsets. Given a Borel measure space $\langle X, \Sigma, \mu\rangle$ and some subset $A \subseteq X$, let $\mathcal{C} \subseteq \Sigma$ be a **cover** of A if $\mathcal{C}$ is countable and $A \subseteq \bigcup \mathcal{C}$. We define the **Lebesgue outer measure** $\lambda^*(A)$ of a subset $A \subseteq X$ as the infimum of $\left\{\sum_{c\in\mathcal{C}}\mu(c) \mid \mathcal{C} \text{ is a cover of } A\right\}$. A set $A \subseteq X$ is **measurable** if it satisfies Carathéodory's criterion: for every $B \subseteq X$ we have $\lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \setminus A)$.

We define the **Lebesgue measure space** $\langle X, \Sigma', \lambda\rangle$ such that Σ' is the set of measurable subsets of X, and we let $\lambda(A) = \lambda^*(A)$ for every $A \in \Sigma'$. Equivalently we could define $\langle X, \Sigma', \lambda\rangle$ such that λ is a complete measure, Σ' is the least σ-algebra with $\Sigma \subseteq \Sigma'$ and $\lambda(A) = \mu(A)$ for all $A \in \Sigma$. $\triangleleft$

In particular, when we talk about the real numbers $\mathbb{R}$, we have the Borel measure μ generated by basic open sets (a, b) with $a \leq b$ having measure $\mu((a, b)) = b - a$, and we have the Lebesgue measure λ that is the completion of μ. We use null to denote the set of Lebesgue null sets of the reals. A subset $A \subseteq \mathbb{R}$ is a Lebesgue null set if and only if for any $\varepsilon > 0$ there is a cover C of A with measure $\mu(C) < \varepsilon$.

Definition 1.2.15 — *Baire category*
Given a topological space X, a subset $A \subseteq X$ is **nowhere dense** if for every $a \in A$ and open neighbourhood U of a there is an open $V \subseteq U$ such that $A \cap V = \varnothing$. The closure of a nowhere dense set is nowhere dense, and the complement of a closed nowhere dense set is open dense. A set is **meagre** if it is the countable union of nowhere dense sets. The complement of a meagre set is called **comeagre**. A comeagre set is the countable intersection of sets that contain an open dense subset. The set of all meagre sets is denoted by meagre.

A topological space is **Baire** if every comeagre set is dense. A subset $A \subseteq X$ of a topological space has the **property of Baire** if there is an open set $U \subseteq X$ such that the symmetric difference $U \triangle A$ is meagre. $\triangleleft$

Theorem 1.2.16 — *Baire Category Theorem*
Every complete metric space is Baire. ◁

As a corollary every Polish space is Baire.

Definition 1.2.17 — *Baire sets*
Let κ be a cardinal and consider the product space ${}^{\kappa}2$, where 2 has the discrete topology. The space ${}^{\kappa}2$ has a basis of clopens given by the sets $U_s = \{f \in {}^{\kappa}2 \mid \forall n \in \mathrm{dom}(s)(f(n) = s(n))\}$ with $s \in \mathrm{Fn}_{\aleph_0}(\kappa, 2)$.

Let $\mathcal{B}({}^{\kappa}2)$ be the smallest σ-algebra containing the basis of clopens. We call a set $X \subseteq {}^{\kappa}2$ a **Baire set** if $X \in \mathcal{B}({}^{\kappa}2)$. Every Baire set has the property of Baire. ◁

Before we move on to forcing, let us mention a couple of *zero-one laws.* A zero-one law for some ideal, such as the null and meagre ideals, is a statement that gives a sufficient condition on a set of reals A to imply that either A or the complement of A is an element of the ideal.

Theorem 1.2.18 — *Rademacher's zero-one law*
If $\overline{a} \in {}^{\omega}\mathbb{R}$ be an infinite sequence of real numbers and let

$$S = \left\{ s \in {}^{\omega}2 \;\middle|\; \textstyle\sum_{n\in\omega}(-1)^{s(n)}a_n \text{ converges} \right\}.$$

If $\sum_{n\in\omega} {a_n}^2$ diverges, then the Lebesgue measure $\mu(S) = 0$ and if it converges $\mu(S) = 1$. ◁

Definition 1.2.19 — *Tail set*
A set $X \subseteq {}^{\omega}\omega$ is a **tail set** if X is closed under $=^*$, that is, for any $f, g \in {}^{\omega}\omega$ for which $f(k) = g(k)$ for almost all $k \in \omega$ we have $f \in X$ if and only if $g \in X$. ◁

Theorem 1.2.20 — *Baire category zero-one law*
If $X \subseteq {}^{\omega}\omega$ is a tail set with the property of Baire, then either $X \in \mathsf{meagre}$ or ${}^{\omega}\omega \setminus X \in \mathsf{meagre}$. ◁

1.3 Forcing

Kurt Gödel showed with his constructible universe L that a model of ZF could be employed to construct a model where every set is transfinite recursively definable. The constructible universe turns out to be a model of ZF itself, and moreover to be a model of both AC and GCH. This proves that ZF itself can not prove that AC or GCH is inconsistent.

In 1963 Paul Cohen [8, 9] showed the other direction, that from a model of ZF one can construct models in which AC or GCH do not hold, and thus that AC and GCH are also not provable in ZF. The technique that Cohen used is called the method of **forcing**.

Definition 1.3.1 — *Forcing poset*
A **forcing poset** is a triple $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ such that $\leq$ is a preorder[1] on $\mathbb{P}$ with maximal element $\mathbb{1}$ such that $\mathbb{P}$ is **atomless**, that is, for any $p \in \mathbb{P}$ there are $q, r \in \mathbb{P}$ such that $q, r \leq p$ and for any $s \in \mathbb{P}$ we have $s \not\leq q$ or $s \not\leq r$. We often only write $\mathbb{P}$ if $\leq$ and $\mathbb{1}$ are clear from context.

The elements of $\mathbb{P}$ are called **conditions**, and if $q \leq p$, then q is called a **stronger** condition than p. Two elements $p, q \in \mathbb{P}$ are **compatible**, denoted $p \,||\, q$, if there is some $r \in \mathbb{P}$ such that $r \leq p, q$. Two elements p, q that are not compatible are **incompatible**, denoted $p \perp q$. An **antichain** is a set $\mathcal{A} \subseteq \mathbb{P}$ such that for any $p, q \in \mathcal{A}$ we have $p \perp q$. $\triangleleft$

Note that we could relax the requirement for a forcing poset to be atomless, but this could imply that a forcing poset is trivial. For example, if $\mathbb{P}$ is finite, then it will follow that forcing with $\mathbb{P}$ does not have any effect.

Definition 1.3.2 — *Generic filter*
Let $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ be a forcing poset. A **filter** on $\mathbb{P}$ is a subset $F \subseteq \mathbb{P}$ such that $\mathbb{1} \in F$, F is upward closed under $\leq$ and any two $p, q \in F$ are compatible. A set $D \subseteq \mathbb{P}$ is **dense** in $\mathbb{P}$ if for every condition $p \in \mathbb{P}$ there is some $q \in D$ such that $q \leq p$. Let $\mathcal{D}$ be a set of dense subsets of $\mathbb{P}$, then a filter G is **generic** with respect to $\mathcal{D}$ if $G \cap D \neq \varnothing$ for all $D \in \mathcal{D}$. $\triangleleft$

Under the assumption that there is a model for ZFC, there is a countable model by the Downward Löwenheim-Skolem Theorem. Such a countable model is isomorphic to a countable transitive model by use of the Mostowski Collapse Lemma. We will use the abbreviation **ctm** to denote a **countable transitive model of** (a part of) ZFC. Forcing starts with a **ground model** of (a part of) ZFC containing some forcing poset $\mathbb{P}$. We will adopt the external view that the ground model is a ctm $\mathcal{M}$ and use the countability of $\mathcal{M}$ to show that in the external model V there exists a filter G of $\mathbb{P}$ that is generic with respect to every dense subset $D \subseteq \mathbb{P}$ that is contained in $\mathcal{M}$.

Lemma 1.3.3
Let $\mathcal{M}$ be a ctm containing a set $\mathbb{P}$ and such that

$$\mathcal{M} \vDash \text{“}\mathbb{P} \text{ is a forcing poset and } p \in \mathbb{P}\text{”}.$$

Let $\mathcal{D}_{\mathcal{M}}$ be the set of dense subsets of $\mathbb{P}$ such that $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$. Then

$$V \vDash \text{“there exists a } \mathcal{D}_{\mathcal{M}}\text{-generic filter } G \subseteq \mathbb{P} \text{ containing } p.\text{”}$$

Furthermore, $V \vDash G \notin \mathcal{M}$. $\triangleleft$

Often, when we have a ctm $\mathcal{M}$ and a poset $\mathbb{P}$ with a generic filter G, we will just call G **generic** for $\mathbb{P}$. We extend this to objects that can be defined from G and from which G can be recovered through the structure of $\mathbb{P}$. For example, if $\mathbb{P}$ is a set of partial functions inversely ordered

[1] We speak of forcing posets, instead of forcing preorders, since from the perspective of forcing it is equivalent to work with preorders directly or to work with the quotient set generated by collapsing the cycles of the preorder.

by inclusion, then we get a function $f = \bigcup G$, from which we can recover G by looking at all conditions $p \in \mathbb{P}$ such that $p \subseteq f$.

Definition 1.3.4 — *Names*
Let $\mathbb{P}$ be a forcing poset in a ctm $\mathcal{M}$. A set $\sigma \in \mathcal{M}$ is recursively defined to be a $\mathbb{P}$-**name** if σ consists of pairs (τ, p) such that τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$, and such that if $(\tau, p), (\tau, p') \in \sigma$, then $p \neq p'$ implies $p \perp p'$. The class of all $\mathbb{P}$-names is denoted $\mathcal{M}^{\mathbb{P}}$. Note that $\mathcal{M}^{\mathbb{P}}$ is a proper class in $\mathcal{M}$ and a countable set in V. Every set $x \in \mathcal{M}$ has a canonical $\mathbb{P}$-name $\check{x} = \{(\check{y}, \mathbb{1}) \mid y \in x\}$. We will usually ignore the distinction between canonical names for elements of $\mathcal{M}$ and elements themselves, that is, we will usually omit the check on canonical names.

If $G \subseteq \mathbb{P}$ is generic, then the **interpretation** of a name σ under G is the set σ_G defined as $\sigma_G = \{\tau_G \mid \exists p \in G((\tau, p) \in \sigma)\}$. We define the **generic extension** $\mathcal{M}[G]$ to be the set of all interpretations of names under G, that is, $\mathcal{M}[G] = \{\sigma_G \mid \sigma \in \mathcal{M}^{\mathbb{P}}\}$. If $x \in \mathcal{M}[G]$, then $\dot{x}$ denotes some name σ such that $\sigma_G = x$. Such σ is not unique, but the choice of σ is irrelevant. $\triangleleft$

It is easy to see that $\mathcal{M} \subseteq \mathcal{M}[G]$, as for any $x \in \mathcal{M}$ the canonical name $\check{x}$ has interpretation $\check{x}_G = x$, because $\mathbb{1} \in G$ for any filter G. Furthermore, the name $\Gamma = \{\langle \check{p}, p\rangle \mid p \in \mathbb{P}\}$ is interpreted as $\Gamma_G = G$, showing that $G \in \mathcal{M}[G]$.

The whole point of forcing is that the set $\mathcal{M}[G]$ forms a model of (a part of) ZFC that inherits properties from the structure of the poset $\mathbb{P}$. For this we have the following definition that gives us a way to describe which formulas hold in the generic extensions for any generic filter containing a condition.

Theorem 1.3.5
If $\mathcal{M}$ is a ctm and $\mathbb{P}$ is a forcing poset with generic filter G, then $\mathcal{M}[G]$ is a ctm, that is, $\mathcal{M}[G]$ is countable (in V), transitive and $\mathcal{M}[G] \vDash$ (a part of) ZFC. $\triangleleft$

Definition 1.3.6 — *Forcing relation*
We define the **forcing relation** $\Vdash$ such that for any $p \in \mathbb{P}$, names $\dot{x}^1, \dots, \dot{x}^n \in \mathcal{M}^{\mathbb{P}}$ and formula $\varphi(\dot{x}^1, \dots, \dot{x}^n)$ we have $p \Vdash \varphi(\dot{x}^1, \dots, \dot{x}^n)$ if and only if $\mathcal{M}[G] \vDash \varphi(\dot{x}^1_G, \dots, \dot{x}^n_G)$ for all generic G with $p \in G$. We say that p **forces** φ when $p \Vdash \varphi$. $\triangleleft$

The proof of Theorem 1.3.5 is technical, and depends on the following two important lemmas:

Lemma 1.3.7 — *Definability lemma*
Let $\mathcal{M}$ be a ctm and $\varphi(x_1, \dots, x_n)$ be a formula in the language $\{\in\}$, then the following set is definable in $\mathcal{M}$:

$$\begin{aligned}\big\{(p, \mathbb{P}, \leq, \mathbb{1}, \theta_1, \dots, \theta_n) \mid (\mathbb{P}, \leq, \mathbb{1}) \in \mathcal{M} \text{ is a forcing poset with } p \in \mathbb{P}\\ \text{and } \theta_1, \dots, \theta_n \in \mathcal{M}^{\mathbb{P}} \text{ and } p \Vdash \varphi(\theta_1, \dots, \theta_n)\big\}\end{aligned} \quad \triangleleft$$

Lemma 1.3.8 — *Truth lemma*
If $\mathcal{M}$ is a ctm containing forcing poset $\mathbb{P}$, and $\mathcal{M}[G]$ is a generic extension, then $\mathcal{M}[G] \vDash \varphi$ if and only if there is a $p \in G$ such that $p \Vdash \varphi$. $\triangleleft$

The details about Theorem 1.3.5 and the above two lemmas can be read in chapter IV.2 of Kunen [20]. The main point to note about these two lemmas is that the forcing relation $\Vdash$ can be defined inside $\mathcal{M}$, and that any formula φ that holds in the generic extension $\mathcal{M}[G]$ is forced by some $p \in G$.

We can prove the following properties of the forcing relation:

Proposition 1.3.9
Let $\mathcal{M}$ be a ctm containing forcing poset $\mathbb{P}$, and let φ, ψ be sentences in the language of set theory with constants for every name in $\mathcal{M}^{\mathbb{P}}$.

- If $\varphi \leftrightarrow \psi$, then $p \Vdash \varphi$ iff $p \Vdash \psi$,
- $p \Vdash \neg\varphi$ iff there exists no $q \leq p$ with $q \Vdash \varphi$,
- $p \Vdash \varphi$ implies $p \nVdash \neg\varphi$,
- $p \Vdash \varphi$ and $q \leq p$ implies $q \Vdash \varphi$,
- $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$,
- $p \Vdash \exists x \varphi(x)$ iff there is some $q \leq p$ and name $\dot{x}$ such that $q \Vdash \varphi(\dot{x})$. ◁

Next, we look at a few techniques to show equivalence between forcing with different posets.

Theorem 1.3.10
If $\mathcal{M}$ is a ctm containing forcing poset $\mathbb{P}$ and G is generic, then for any model $\mathcal{N} \vDash \mathsf{ZFC}$ such that $\mathcal{M} \subseteq \mathcal{N}$ and $G \in \mathcal{N}$, we have $\mathcal{M}[G] \subseteq \mathcal{N}$. That is, $\mathcal{M}[G]$ is the smallest extension of $\mathcal{M}$ containing G. ◁

Lemma 1.3.11
If $\mathbb{P} \subseteq \mathbb{Q}$ are posets, and $\mathbb{P}$ is dense in $\mathbb{Q}$ and $G \subseteq \mathbb{P}$ is a filter with $H = \{q \in \mathbb{Q} \mid \exists p \in G(p \leq q)\}$, then G is generic for $\mathbb{P}$ whenever H is generic for $\mathbb{Q}$ and vice versa. ◁

Definition 1.3.12 — *Dense embedding*
A map $\iota : \mathbb{P} \hookrightarrow \mathbb{Q}$ of posets is a **dense embedding** if for any $p, q \in \mathbb{P}$ we have

- $\iota(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$
- $p \leq_{\mathbb{P}} q$ implies $\iota(p) \leq_{\mathbb{Q}} \iota(q)$
- $p \perp_{\mathbb{P}} q$ if and only if $\iota(p) \perp_{\mathbb{Q}} \iota(q)$
- $\iota(\mathbb{P})$ is dense in $\mathbb{Q}$.

Here we used subscripts to make the distinction between the poset structure in $\mathbb{P}$ and in $\mathbb{Q}$. ◁

Theorem 1.3.13
If $\iota : \mathbb{P} \hookrightarrow \mathbb{Q}$ is a dense embedding, then $\mathcal{M}[G] = \mathcal{M}[\uparrow\iota(G)]$ for any G generic for $\mathbb{P}$ and $\mathcal{M}[\iota^{-1}(H)] = \mathcal{M}[H]$ for any H generic for $\mathbb{Q}$. ◁

It follows that the generic extensions after forcing with $\mathbb{P}$ are the same as the generic extensions after forcing with $\mathbb{Q}$, thus $\mathbb{P}$ and $\mathbb{Q}$ are equivalent for forcing.

For some posets, forcing with them will give us reals in the generic extension that were not present in the ground model. Different posets will add reals with different properties, as we will see later on.

Adding reals with a certain property can change the properties of sets defined on the reals, such as the null and meagre ideals. In particular, they could alter the cardinality of cardinal characteristics. We do not want the forcing posets to change *too much* about the ground model, however. It is, for our purposes, especially essential that we do not change which ordinals are cardinals.

Definition 1.3.14 — *Chain condition*

A poset $\mathbb{P}$ has the κ-chain condition if $\mathbb{P}$ contains no antichain of cardinality κ. If $\kappa = \aleph_1$ we call this the countable chain condition (ccc). $\triangleleft$

Theorem 1.3.15

If $\mathbb{P}$ is ccc, then cardinals and cofinalities are preserved by forcing with $\mathbb{P}$. That is, if $\mathbb{P} \in \mathcal{M}$ and G is generic for $\mathbb{P}$, then an ordinal $\alpha \in \mathcal{M}$ is a cardinal in $\mathcal{M}$ if and only if α is a cardinal in $\mathcal{M}[G]$, and similarly the cofinality of α is the same in $\mathcal{M}$ and $\mathcal{M}[G]$. $\triangleleft$

A useful tool to prove that posets are ccc is the following lemma, due to Nikolai Shanin [33]:

Lemma 1.3.16 — *Δ-lemma*

If κ is a regular uncountable cardinal and $\mathcal{X} = \{X_\alpha \mid \alpha < \kappa\}$ is a family of distinct finite sets, then there is a subset $\mathcal{Y} \subseteq \mathcal{X}$ with $|\mathcal{Y}| = \kappa$ and a set R such that for any $X_\alpha, X_\beta \in \mathcal{Y}$ with $\alpha \neq \beta$ we have $X_\alpha \cap X_\beta = R$. $\triangleleft$

Another condition that preserves cardinals and cofinalities under forcing extensions is the following.

Definition 1.3.17 — *σ-centred*

A subset $A \subseteq \mathbb{P}$ is **centred** if for any finite subset $\{p_0, \dots, p_n\} \subseteq A$ there is some $q \in \mathbb{P}$ with $q \leq p_i$ for each $0 \leq i \leq n$. A poset $\mathbb{P}$ is **σ-centred** if $\mathbb{P} = \bigcup_{n\in\omega} \mathbb{P}_n$ with each $\mathbb{P}_n$ being centred in $\mathbb{P}$. $\triangleleft$

It is easy to see that every σ-centred poset is ccc, since no two elements of an antichain can lie in the same $\mathbb{P}_n$.

Cohen forcing

Cohen forcing is the forcing used by Paul Cohen to prove the independence of CH from the axioms of ZFC. The idea is to start with a model of ZFC+CH (for example Gödel's constructible universe $L \vDash \mathsf{ZFC} + \mathsf{CH}$) and increase the cardinality of the continuum by generically adding a large number of new reals.

We will see a real number as an element of ${}^{\omega}2$. The easiest way to see that a forcing will add a certain object is by having the conditions be an approximation of such an object. We therefore consider the set of finite approximations of a potential new real as our poset.

Definition 1.3.18 — *Cohen forcing*
The **Cohen forcing** $\langle \mathbb{C}(\kappa), \leq_{\mathbb{C}(\kappa)}, \mathbb{1}_{\mathbb{C}(\kappa)} \rangle$ has the poset with elements $\mathbb{C}(\kappa) = \mathrm{Fn}_{\aleph_0}(\kappa, 2)$ of finite partial functions $p : \kappa \nrightarrow 2$ and is ordered inversely by inclusion, i.e. $q \leq p$ if $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and $q(n) = p(n)$ for all $n \in \mathrm{dom}(p)$. This gives us that $\mathbb{1}_{\mathbb{C}(\kappa)} = \varnothing$. ◁

If $G \subseteq \mathbb{C}(\kappa)$ is a generic filter, then $\bigcup G$ is a function $\kappa \to 2$. This is easy to see after noting first that all elements of G are comparable, and thus must agree on the values in their domain, and second that for any $\alpha < \kappa$ the set of conditions $p \in \mathbb{C}(\kappa)$ with $\alpha \in \mathrm{dom}(p)$ are dense.

Slightly less clear is that if $\mathcal{M} \vDash \mathsf{ZFC}$ and $\mathbb{C}(\kappa)$ is the Cohen poset of $\mathcal{M}$ with a generic filter G, then $\bigcup G \notin \mathcal{M}$. Stronger even, for any limit ordinal $\alpha < \kappa$ we have that $\bigcup G \upharpoonright [\alpha, \alpha + \omega)$ is a function not in $\mathcal{M}$. To see this, take any real $s \in {}^{\omega}2 \cap \mathcal{M}$ and translate s to the function $s_\alpha : [\alpha, \alpha + \omega) \to 2$ with $\alpha + n \mapsto s(n)$. The set of conditions $p \in \mathbb{C}(\kappa)$ with $\alpha + n \in \mathrm{dom}(p)$ such that $p(\alpha + n) \neq s_\alpha(\alpha + n)$ for some $n \in \omega$ form a dense set in $\mathbb{C}(\kappa)$, hence by genericity G contains for any real in $\mathcal{M}$ a condition that disagrees with it. This implies the claim that $\bigcup G \upharpoonright [\alpha, \alpha + \omega)$ is a new real.

Finally if we have two distinct limit ordinals $\alpha, \beta < \kappa$, then $\bigcup G \upharpoonright [\alpha, \alpha+\omega)$ and $\bigcup G \upharpoonright [\beta, \beta+\omega)$ describe two different reals. Once again, we have a density argument: the set of conditions $p \in \mathbb{C}(\kappa)$ such that $p(\alpha + n) \neq p(\beta + n)$ for some $n \in \omega$ is dense in $\mathbb{C}(\kappa)$, and thus G contains such p for any distinct limit ordinals $\alpha, \beta < \kappa$.

We therefore see that forcing with $\mathbb{C}(\kappa)$ adds κ many new reals. We call such reals **Cohen reals**, and often forcing with $\mathbb{C}(\kappa)$ is described as *"adding κ many Cohen reals."*

Theorem 1.3.19
Let $\mathcal{M} \vDash \mathsf{GCH}$ be a ctm, let $\kappa > \aleph_1$ be regular and let G be a generic filter for $\mathbb{C}(\kappa)$, then $\mathcal{M}[G] \vDash \mathfrak{c} = \kappa$. ◁

Proof. First we observe that $\mathbb{C}(\kappa)$ is ccc. This is a consequence of the Δ-lemma: if we had an uncountable antichain $\mathcal{A} = \{p_\alpha \mid \alpha < \omega_1\} \subseteq \mathbb{C}(\kappa)$, then $\mathrm{dom}(p_\alpha)$ is a finite set for each $\alpha < \omega_1$, thus the Δ-lemma gives us some uncountable set of conditions $\mathcal{A}' \subseteq \mathcal{A}$ and some set R such that $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = R$ for every distinct $\alpha, \beta < \omega_1$. But there are only finitely many partial functions $R \to 2$, thus $\mathcal{A}'$ cannot be an antichain.

As $\mathbb{C}(\kappa)$ is ccc it preserves the cardinality of κ. There is an injection between κ and the reals $\bigcup G \upharpoonright [\alpha, \alpha + \omega)$ defined by G, so we see that $\mathcal{M}[G] \vDash 2^{\aleph_0} \geq \kappa > \aleph_1$. We could furthermore show that $\mathcal{M}[G] \vDash 2^{\aleph_0} \leq \kappa$, since every set $x \in \mathcal{M}[G]$ is named by some $\dot{x} \in \mathcal{M}^{\mathbb{P}}$. In particular, if $s \in \mathcal{P}(\omega)$ is a real in $\mathcal{M}[G]$, then it has a name $\dot{s}$ of the form $\{\{\check{n}\} \times A_n \mid n \in \omega\}$, where each $\check{n}$ is the canonical name for $n \in \omega$ and each A_n is an antichain. Since $|\mathbb{C}(\kappa)| = \kappa$ and each antichain is countable, there are at most $\kappa^{\aleph_0}$ antichains, and hence there are at most $\kappa^{\aleph_0 \cdot \aleph_0} = \kappa^{\aleph_0}$ names for reals in $\mathcal{M}[G]$. Since $\mathcal{M} \vDash \mathsf{GCH}$ and κ is regular we have $\kappa^{\aleph_0} = \kappa$. □

The model $\mathcal{M}[G]$ from the above theorem with $\kappa = \aleph_2$ is called the **Cohen model**.

We will give three alternative forcings that are equivalent to Cohen forcing. First, forcing with a countable forcing poset is the same as adding a single Cohen real.

Theorem 1.3.20
If $\mathbb{P}$ is a countable forcing poset, then $\mathbb{P}$ is forcing equivalent to $\mathbb{C}(\aleph_0)$. $\triangleleft$

Proof. Fix an enumeration of the elements of $\mathbb{P} = \{p_n \mid n \in \omega\}$. Let $\mathbb{C}' \subseteq \mathbb{C}(\aleph_0)$ with as set of conditions $\omega^{<\omega}$. It is easy to see that $\mathbb{C}'$ is dense in $\mathbb{C}(\aleph_0)$. We create a dense embedding $\iota : \mathbb{C}' \hookrightarrow \mathbb{P}$ recursively as follows:

Set $\iota(\mathbb{1}_{\mathbb{C}'}) = \mathbb{1}_{\mathbb{P}}$. At stage n, assume we defined $\iota(f)$ for all $f = (f_0, \dots, f_{n-1}) \in \omega^n$. For each $f \in \omega^n$ let $A_f \subseteq \mathbb{P}$ be an antichain that is maximal in that for all $q \in A_f$ we have $q < \iota(f)$ and $q < p_k$ for all $k \leq n$. A_f exists, since $\mathbb{P}$ is atomless, and A_f is countable since $\mathbb{P}$ is countable, thus we can enumerate $A_f = \{q_k \mid k \in \omega\}$. We then define $\iota : (f_0, f_1, \dots, f_{n-1}, k) \mapsto q_k$ for all $k \in \omega$. $\square$

Second, Cohen forcing is closely related to Baire category. Let us first mention how we can *code* meagre sets using reals. The following property is a basic consequence of the closure of nowhere dense sets being nowhere dense.

Proposition 1.3.21
If M is a meagre set, then there is some $M' \supseteq M$ such that M' is meagre and an F_σ set. $\triangleleft$

Fix an enumeration $\{I_n \mid n \in \omega\}$ of all open intervals with rational endpoints, then every real $f \in {}^{\omega\times\omega}\omega$ codes an F_σ set $\mathbb{R} \setminus \bigcap_{m\in\omega} \bigcup_{n\in\omega} I_{f(n,m)}$. Since every open set of reals is the union of countably many open sets with rational endpoints, it follows that every F_σ set is coded by some real.

That Cohen forcing is related to meagre sets becomes especially clear when we use the following alternative definition of the forcing.

Theorem 1.3.22
Let $\mathcal{B}({}^{\omega}2)$ be the set of Baire sets of the product space ${}^{\kappa}2$, then we can define the quotient set $\mathbb{M}(\kappa) = (\mathcal{B}({}^{\kappa}2) \setminus \mathsf{meagre})/\mathsf{meagre}$ given by nonmeagre sets $X, Y \in \mathcal{B}({}^{\kappa}2)$ being equivalent if $X \triangle Y \in \mathsf{meagre}$. We define an order on $\mathbb{M}(\kappa)$ as $[Y] \leq [X]$ if and only if $X \setminus Y \in \mathsf{meagre}$. $\mathbb{M}(\kappa)$ is forcing equivalent to $\mathbb{C}(\kappa)$. $\triangleleft$

Proof. The space ${}^{\kappa}2$ has a basis of clopens $U_s = \{f \in {}^{\kappa}2 \mid s \subseteq f\}$ for $s \in \mathrm{Fn}_{\aleph_0}(\kappa, 2)$. Every Baire set has the property of Baire, so it follows that the equivalence classes $\{[U_f] \mid f \in \mathrm{Fn}_{\aleph_0}(\kappa, 2)\}$ form a dense subset of $\mathbb{M}(\kappa)$. $\square$

Since every Baire set has the property of Baire, we see that every $[X] \in \mathbb{M}(\kappa)$ contains a nonempty open set $U \in [X]$. Since nonempty open sets are nonmeagre, and the difference between two distinct nonempty open sets is open, it follows that this open set is unique for each equivalence class, thus we have yet another equivalent way of looking at Cohen forcing:

Theorem 1.3.23
Let $\mathbb{O}(\kappa)$ be the set of nonempty open subsets of the product space ${}^{\kappa}2$. We define an order on $\mathbb{O}(\kappa)$ as $X \leq Y$ if and only if $X \subseteq Y$. Then $\mathbb{O}(\kappa)$ is forcing equivalent to $\mathbb{C}(\kappa)$. ◁

We are now ready to give a defining property of Cohen reals.

Theorem 1.3.24
A real $r \in \mathbb{R} \cap \mathcal{M}[G]$ is a Cohen real if and only if r is not contained in any meagre set that is coded by a real from the ground model. ◁

> *Proof.* We show only one of the directions. To see that Cohen reals fall outside every meagre set coded in the ground model, let $X = \bigcup X_i$ be a meagre set coded in the ground model and let each X_i be nowhere dense. Let $\overline{X}_i$ be the closure of X_i, then $\overline{X}_i$ is nowhere dense. Take a condition $U \in \mathbb{O}(\kappa)$, then $U \setminus \overline{X}_i$ is open, and it is nonempty since no superset of U is nowhere dense. Therefore the set of conditions that do not intersect X_i is dense in $\mathbb{O}(\kappa)$. It follows that if G is generic for $\mathbb{O}(\kappa)$, then G contains an open set that is disjoint from X_i for any i, and thus the real $r = \bigcap G$ is not contained in X. □

Random forcing

Random forcing is a forcing that is similar to the definition of Cohen forcing as in Theorem 1.3.22, but uses the ideal of Lebesgue null sets instead of meagre sets. It was introduced by Robert Solovay to show that it is consistent with ZF that every subset of $\mathbb{R}$ is Lebesgue measurable.

Definition 1.3.25 — *Random forcing*
The **random forcing** $\langle \mathbb{B}(\kappa), \leq_{\mathbb{B}(\kappa)}, \mathbb{1}_{\mathbb{B}(\kappa)} \rangle$ has the poset with elements $\mathbb{B}(\kappa) = (\mathcal{B}({}^{\kappa}2) \setminus \mathsf{null})/\mathsf{null}$, where $\mathcal{B}({}^{\kappa}2) \setminus \mathsf{null}$ is the set of Baire sets of ${}^{\kappa}2$ with positive measure and where $X, Y \in \mathcal{B}({}^{\kappa}2)$ are equivalent if $X \triangle Y \in \mathsf{null}$. We order $\mathbb{B}(\kappa)$ by letting $[Y] \leq_{\mathbb{B}(\kappa)} [X]$ if $X \setminus Y \in \mathsf{null}$. ◁

Much like Cohen forcing, a generic filter $G \subseteq \mathbb{B}(\kappa)$ defines a function $r : \kappa \to 2$ for which each $r \restriction [\alpha, \alpha + \omega)$ with $\alpha < \kappa$ limit is the translation of a generic real. Such reals are called **random reals**, and forcing with $\mathbb{B}(\kappa)$ is described as *"adding κ many random reals."*

Often it is convenient to work with the following equivalent forcing.

Theorem 1.3.26
Let $\mathbb{N}(\kappa)$ be the set of compact subsets of ${}^{\kappa}2$ with positive measure, ordered by inclusion, that is, if $X, Y \subseteq {}^{\kappa}2$ are compact, then X is a stronger condition than Y if $X \subseteq Y$. Then $\mathbb{N}(\kappa)$ is forcing equivalent to $\mathbb{B}(\kappa)$. ◁

> *Proof.* The dense embedding is given by mapping $X \in \mathbb{N}(\kappa)$ to $[X] \in \mathbb{B}(\kappa)$. This is dense, since the Lebesgue measure on ${}^{\kappa}2$ has the regularity property that for any measurable set $X \subseteq {}^{\kappa}2$ and $\varepsilon > 0$ there is some open set O and compact set C such

that $C \subseteq X \subseteq O$ and $\mu(O \setminus C) < \varepsilon$. In particular for any Baire set $X \in \mathcal{B}({}^{\kappa}2)$ with positive measure there is some compact set $C \subseteq X$ of positive measure, which shows that the abovementioned embedding is dense. $\square$

We also have a characterisation of random reals regarding null sets from the ground model that is similar to the case with Cohen reals.

Proposition 1.3.27
If N is a null set, then there is some $N' \supseteq N$ such that N' is null and an F_σ set. $\triangleleft$

Theorem 1.3.28
A real $r \in \mathbb{R} \cap \mathcal{M}[G]$ is a random real if and only if r is not contained in any null set that is coded by a real from the ground model. $\triangleleft$

Proof. Again, we treat just one of the directions. To see that random reals fall outside every null set coded in the ground model, let X be a null set coded in the ground model and let $C \in \mathbb{N}(\kappa)$ be a compact set, then $C \setminus X$ is measurable, and thus there is some compact set $C' \subseteq C \setminus X$ with positive measure, hence $C' \in \mathbb{N}(\kappa)$. This shows that a dense subset of $\mathbb{N}(\kappa)$ is disjoint from X, and thus if G is generic for $\mathbb{N}(\kappa)$, then G contains a set disjoint from X for any null set coded in the ground model. The set $r = \bigcap G$ is a generic function from $\kappa \to 2$ that lies in no null set coded in the ground model. $\square$

In a way that is similar to the Cohen model, $\mathbb{B}(\kappa)$ can be used to get models that violate the continuum hypothesis.

Theorem 1.3.29
Let $\mathcal{M} \models \mathsf{GCH}$ be a ctm, let $\kappa > \aleph_1$ be regular and let G be a generic filter for $\mathbb{B}(\kappa)$, then $\mathcal{M}[G] \models \mathfrak{c} = \kappa$. $\triangleleft$

Proof. First, note that $\mathbb{B}(\kappa)$ is ccc as a consequence of the σ-additivity of measure. If we had an uncountable antichain $A \subseteq \mathbb{B}(\kappa)$ there must be some $p \in A$ such that there is a countably infinite set $B \subseteq A$ with $0 < \mu(p) \leq \mu(q)$ for every $q \in B$. But as all these q are mutually disjoint (up to a null set) due to B being an antichain, we would get that $\mu(\bigcup B) = \sum_{q \in B} \mu(q) = \infty$, which is impossible, since $\mu({}^{\kappa}2) = 1$.

As $\mathbb{B}(\kappa)$ is ccc, it preserves the cardinality of κ. That $2^{\aleph_0} \geq \kappa$ follows by the generic function $\kappa \to 2$ that is added by G having the property that each $r \upharpoonright [\alpha, \alpha + \omega)$ defines a distinct generic real. The argument that $2^{\aleph_0} \leq \kappa$ is exactly as with the proof of Theorem 1.3.19. $\square$

In case $\kappa = \aleph_2$ in the above theorem, we call the resulting model $\mathcal{M}[G]$ the **random model**.

Finally we will state a theorem without proof that gives us a condition for *not* adding random reals with a forcing.

Theorem 1.3.30
If $\mathcal{M}$ is a ctm, $\mathbb{P}$ is a σ-centred forcing poset and G is generic for $\mathbb{P}$, then no $r \in {}^\omega\omega \cap \mathcal{M}[G]$ is random over $\mathcal{M}$. ◁

Iterated forcing

If we have a ctm $\mathcal{M}_0$ and a forcing poset $\mathbb{P}_0$ with some generic filter G_0 for $\mathbb{P}_0$, then forcing with $\mathbb{P}_0$ will result in a model $\mathcal{M}_0[G_0] = \mathcal{M}_1$. Of course, there's nothing stopping us from continuing this process and take some forcing poset $\mathbb{P}_1$ in $\mathcal{M}_1$ with some generic filter G_1 for $\mathbb{P}_1$ to get a model $\mathcal{M}_1[G_1] = \mathcal{M}_2$, and so on. This process is called **iterated forcing**.

We have to take care with the model in which a forcing poset is interpreted. For example, the poset $\mathbb{P}_1$ that is mentioned above is an element of $\mathcal{M}_1$, and might very well not exist in $\mathcal{M}_0$. This means that we cannot force with $\mathbb{P}_1$ directly over $\mathcal{M}_0$. On the other hand, $\mathbb{P}_0$ is a poset defined in $\mathcal{M}_0$, thus it could be that defining the same set in $\mathcal{M}_1$ is different from $\mathbb{P}_0$. We therefore give the following definition of iterated forcing, that is a little more careful about these points.

Definition 1.3.31 — *Iterated forcing*
If $\mathbb{P}$ is a forcing poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a forcing poset, that is $\mathbb{1}_\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is a forcing poset", then we define the **two-step iteration** $\mathbb{P} * \dot{\mathbb{Q}}$ as the forcing poset with conditions $\langle p, \dot{q}\rangle$ with $p \in \mathbb{P}$ and $\mathbb{1}_\mathbb{P} \Vdash \dot{q} \in \dot{\mathbb{Q}}$, and order $\langle p', \dot{q}'\rangle \leq \langle p, \dot{q}\rangle$ if $p' \leq_\mathbb{P} p$ and $p' \Vdash \dot{q}' \leq_{\dot{\mathbb{Q}}} q$.

For an ordinal γ, we define recursively the posets $\mathbb{P}_\alpha$ for $1 \leq \alpha \leq \gamma$ with as conditions sequences of length α. If $p \in \mathbb{P}_\gamma$, then $p \restriction \alpha$ is an element of the poset $\mathbb{P}_\alpha$ defined recursively as follows:

- $\mathbb{P}_1$ is the set of singleton sequences $\langle q\rangle$, where $q \in \mathbb{Q}_0$ is a condition of a forcing poset $\mathbb{Q}_0$. The order on $\mathbb{P}_1$ is defined as $\langle q'\rangle \leq \langle q\rangle$ if $q' \leq_{\mathbb{Q}_0} q$.
- If $\alpha = \beta + 1$, then $\mathbb{P}_\alpha$ is forcing equivalent to $\mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$, where $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a forcing poset. We let $p \in \mathbb{P}_\beta$ if and only if $\langle p \restriction \beta, p(\alpha)\rangle \in \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$.
- For α limit, let X_α be the set of α-sequences p such that $p \restriction \beta \in \mathbb{P}_\beta$ for each $\beta < \alpha$. The **support** of $p \in X_\alpha$ is the set $\operatorname{spt}(p) = \left\{\beta < \alpha \;\middle|\; p(\beta) \neq \mathbb{1}_{\dot{\mathbb{Q}}_\beta}\right\}$. Let I be an ideal on α such that $[\alpha]^{<\aleph_0} \subseteq I$, then $\mathbb{P}_\alpha$ is I-supported if $\mathbb{P}_\alpha = \{p \in X_\alpha \mid \operatorname{spt}(p) \in I\}$. The order on $\mathbb{P}_\alpha$ is defined as $p' \leq_{\mathbb{P}_\alpha} p$ if $p' \restriction \beta \leq_{\mathbb{P}_\beta} p \restriction \beta$ for every $\beta < \alpha$.

We will denote $\mathbb{P}_\gamma$ as $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \;\middle|\; \alpha < \gamma\right\rangle$. We will denote $\mathbb{P}_\alpha$ as $\mathbb{P}_\gamma \restriction \alpha$ and $\dot{\mathbb{Q}}_\alpha$ as $\mathbb{P}_\gamma(\alpha)$. If G is a generic filter for $\mathbb{P}_\gamma$, then $G_\alpha = \{p \restriction \alpha \mid p \in G\}$ is a generic filter for $\mathbb{P}_\alpha$ and $G(\alpha)$ is a generic filter for $\dot{\mathbb{Q}}_\alpha$.

We call $\mathbb{P}_\gamma$ a **finite support iteration** if $\mathbb{P}_\alpha$ is $[\alpha]^{<\aleph_0}$-supported for each limit $\alpha \leq \gamma$, or in other words, if the support of each condition in $\mathbb{P}_\alpha$ is finite. $\mathbb{P}_\gamma$ is a **countable support iteration** if $\mathbb{P}_\alpha$ is $[\alpha]^{<\aleph_1}$-supported for each limit $\alpha \leq \gamma$, that is, if the support of each condition in $\mathbb{P}_\alpha$ is countable. ◁

We will use iterated forcing in the following way. First, we let $\mathbb{P}$ be a forcing poset that will add a generic object G_1 to the ground model $\mathcal{M}$ such that G_1 has a nice property over all the ground model sets. The problem is that $\mathbb{P}$ might also add new *"spoiler"* sets to the model for which G_1 does not have this nice property, thus we need to force with $\mathbb{P}$ *again* in the extension to get a new generic set G_2 with the nice property in the extension of the extension. We could repeat this process transfinitely many times, until we could show that at some limit stage no *spoiler* sets are added. We could then see that we end up with a model where for any set we have some G_α that has the nice property.

Example 1.3.32
For example, the nice property could be that the generic set is a real that does not belong to any meagre set from the ground model. This is what the Cohen forcing $\mathbb{C}(\omega)$ does. However, after forcing with $\mathbb{C}(\omega)$, we have new meagre sets in the extension.

By doing a finite support forcing $\mathbb{P}_{\omega_2}$ of $\mathbb{C}(\omega)$ of length ω_2, we can show that the ω_2-th step does not add any new reals, and thus that for every meagre set M in the extension $\mathcal{M}[G_{\omega_2}]$ it is contained in a meagre set M' coded by a real, and therefore there is some $\alpha < \omega_2$ for which M' is already present in $\mathcal{M}[G_\alpha]$. Subsequently, in $\mathcal{M}[G_{\alpha+1}]$ we add a Cohen real that lies outside M', thus also outside of M, therefore for any collection of $\aleph_1$ meagre sets, we can find some Cohen real that lies outside all of them. $\triangleleft$

In fact, it is not difficult to show that the iteration described above is equivalent to forcing with $\mathbb{C}(\aleph_2)$, due to the fact that we can send a finite partial function $f : \omega_2 \nrightarrow 2$ to a condition $p \in \mathbb{P}_{\omega_2}$ as follows: note that $p(\alpha) : \omega \nrightarrow 2$ is a finite partial function, then let $p(\alpha)(n) = f(\alpha + n)$ for all $\alpha + n$ on which f is defined. It is clear that $\mathrm{spt}(p)$ is finite, since f is only defined on finitely many values.

Note that an iteration of countable length is not sufficient to make sure the limit step does not add reals. We really do need an iteration of length ω_2, since countable limit steps will add reals by the following theorem.

Theorem 1.3.33
If $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \;\middle|\; \alpha < \kappa \right\rangle$ is a finite support iteration of forcings of length κ, then $\mathbb{P}_\gamma$ adds a Cohen real for every γ with $\mathrm{cf}(\gamma) = \omega$. $\triangleleft$

On the other hand, the following theorem states that at uncountable limit steps no reals are added, thus the example does work for an iteration of length ω_2.

Theorem 1.3.34
If $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \;\middle|\; \alpha < \kappa \right\rangle$ is a finite support iteration of forcings of length κ, and G_γ is generic for $\mathbb{P}_\gamma$ with $\gamma \leq \kappa$ of uncountable cofinality, then for any real $r \in {}^\omega\omega \cap \mathcal{M}[G_\gamma]$ there is some $\alpha < \gamma$ such that $r \in \mathcal{M}[G_\alpha]$. $\triangleleft$

We mention one last very nice property of finite support iterations, which is that they preserve the countable chain condition.

Theorem 1.3.35
If $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \;\middle|\; \alpha < \kappa \right\rangle$ is a finite support iteration of forcings and $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is ccc" for each $\alpha < \kappa$, then $\mathbb{P}_\kappa$ is ccc. $\lhd$

As a consequence, a finite support iteration of ccc forcings will preserve cardinalities and cofinalities. Unfortunately the same does not hold for countable support iterations.

Chapter 2

Cardinal Characteristics

Using tools from combinatorics, analysis, topology or measure theory we can describe plenty of interesting subsets of the continuum. If such a subset of the continuum can be proved to be uncountable from its properties, we call it a **cardinal characteristic of the continuum**. Particularly interesting is that the explicit cardinality of many of such cardinal characteristics is independent of ZFC.

By the results from forcing we know that it is consistent that the continuum hypothesis fails, and thus that $\aleph_1 < \mathfrak{c}$ is consistent. The question then becomes what we can say about the cardinality of such subsets of the continuum based on their properties and the relation between the cardinalities of different cardinal characteristics compared to one another. In this chapter we will meet a few of them and show some of the combinatorial relationships between them.

Most of the proofs and theory from this chapter are from Blass' chapter in the Handbook of Set Theory [3].

2.1 Cardinal Characteristics

The first two cardinal characteristics are related to the bounding of functions.

Definition 2.1.1 — *Bounding & dominating numbers*
Let $f, g : \omega \to \omega$, then we say that f **dominates** g or that g is **bounded** by f, denoted as $f \geq^* g$, if $f(n) \geq g(n)$ for all except for finitely many $n \in \omega$.

A set $D \subseteq {}^{\omega}\omega$ is a **dominating set** if for every $g \in {}^{\omega}\omega$ there exists an $f \in D$ such that f dominates g. A set $B \subseteq {}^{\omega}\omega$ is an **unbounded set** if there exists no $g \in {}^{\omega}\omega$ such that every $f \in B$ is bounded by g.

The **dominating number** $\mathfrak{d}$ is the least cardinality of a set $D \subseteq {}^{\omega}\omega$ such that D is dominating. The **bounding number** $\mathfrak{b}$ is the least cardinality of a set $B \subseteq {}^{\omega}\omega$ such that B is unbounded. $\triangleleft$

The cardinal characteristics $\mathfrak{b}$ and $\mathfrak{d}$ are closely related to each other, and by the following theorem we see that the least cardinality of a dominating set of functions has to be larger than the least cardinality of an unbounded set of functions.

Theorem 2.1.2
$\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$. ◁

Proof. Let $B \subseteq {}^\omega\omega$ be countable and enumerate B as $\{f_i \mid i \in \omega\}$. Define $g \in {}^\omega\omega$ as $g(n) = \max\{f_i(n) \mid i \leq n\}$, then g dominates all of B, showing that B is not unbounded.

Let $D \subseteq {}^\omega\omega$ be a set that is not unbounded, then there is $g \in {}^\omega\omega$ such that $f \leq^* g$ for all $f \in D$. Define $g^+ : n \mapsto g(n) + 1$, then $f <^* g$, that is, $f(n) < g(n)$ for all but finitely many $n \in \omega$. But then g^+ is not dominated by any $f \in D$, hence D is not a dominating set.

Clearly the set ${}^\omega\omega$ itself is a dominating set, since any $f \in {}^\omega\omega$ is dominated by $f^+ : n \mapsto f(n) + 1$. □

Next, we look at two cardinal invariants that arise from partitioning infinite subsets of ω into two infinite sets.

Definition 2.1.3 — *Splitting & reaping numbers*
Let $x, y \in [\omega]^\omega$, then we say that y **splits** x when both $x \cap y$ and $x \setminus y$ are infinite. A set $S \subseteq [\omega]^\omega$ is a **splitting set** if every $x \in [\omega]^\omega$ is split by some $s \in S$. A set $R \subseteq [\omega]^\omega$ is a **reaping set** if there exists no $x \in [\omega]^\omega$ such that every $r \in R$ is split by x.

The **splitting number** $\mathfrak{s}$ is the least cardinality of a set $S \subseteq [\omega]^\omega$ such that S is splitting. The **reaping number** $\mathfrak{r}$ is the least cardinality of a set $R \subseteq [\omega]^\omega$ such that R is reaping. ◁

We can relate the size of $\mathfrak{s}$ to $\mathfrak{d}$ and the size of $\mathfrak{r}$ to $\mathfrak{b}$, as is shown in the next two theorems.

Theorem 2.1.4
$\aleph_1 \leq \mathfrak{s} \leq \mathfrak{d}$. ◁

Proof. Let $S \subseteq [\omega]^\omega$ be countable and enumerate S as $\{s_i \mid i \in \omega\}$. We build a sequence of sets t_i such that $t_0 = s_0$, and for each i we have $t_{i+1} = t_i \cap s_i$ if $t_i \cap s_i$ is infinite, and $t_{i+1} = t_i \setminus s_i$ otherwise. It follows that each t_i is infinite. Let $a_0 \in t_0$, and for each i let $a_{i+1} \in t_{i+1} \setminus a_i$. We then have a strictly increasing sequence $\langle a_i \mid i \in \omega\rangle$. The set $A = \{a_i \mid i \in \omega\}$ is not split by any s_i: if $t_{i+1} = t_i \cap s_i$, then all a_j with $j > i$ are elements of s_i, while if $t_{i+1} = t_i \setminus s_i$, then no a_j with $j > i$ is an element of s_i.

Let $D \subseteq {}^\omega\omega$ be a dominating set, then we can assume without loss of generality that $D \subseteq {\uparrow}({}^\omega\omega)$, that is, each function in D is strictly increasing. For each $g \in {\uparrow}({}^\omega\omega)$ define a set $s_g = \big\{m \in \omega \mid \exists n(m \in \big[g^{2n}(0), g^{2n+1}(0)\big))\big\}$. For $x \in [\omega]^\omega$ let $f_x \in {\uparrow}({}^\omega\omega)$

be such that $\operatorname{ran}(f_x) = x$, that is f_x is a bijection from ω to x. Let $g \in D$ dominate f_x, then we will see that s_g splits x. Take $k \in \omega$ be such that $f_x(n) \leq g(n)$ for all $n > k$, and fix an $n \in \omega$ such that $g^n(0) > k$. Then $g^n(0) \leq f_x(g^n(0)) \leq g^{n+1}(0)$, where the first inequality holds by f_x being increasing and the second holds by g bounding f_x above k, in particular at $g^n(0) > k$. Since $f_x(g^n(0)) \in x$, we see that there is an $a \in x \cap \left[g^n(0), g^{n+1}(0)\right)$ for every large enough $n \in \omega$. If n is even, then $a \in s_g$, and if n is odd, then $a \notin s_g$, so both $x \cap s_g$ and $x \setminus s_g$ are infinite. $\mathfrak{s} \leq \mathfrak{d}$ follows since both maps $g \mapsto s_g$ and $x \mapsto f_x$ are injective. $\square$

Theorem 2.1.5

$\mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{c}$. $\triangleleft$

Proof. Clearly the set $[\omega]^\omega$ itself is a reaping set, as no $x \in [\omega]^\omega$ splits itself, so $\mathfrak{r} \leq \mathfrak{c}$.

The argument for $\mathfrak{b} \leq \mathfrak{r}$ uses the same injective maps $g \mapsto s_g$ and $x \mapsto f_x$ as in the previous proof. Suppose $R \subseteq [\omega]^\omega$ has cardinality $|R| < \mathfrak{b}$, and define $B = \{f_x \in {}^\omega\omega \mid x \in R\}$, then B is not unbounded. Let $g \in {}^\omega\omega$ dominate all $f_x \in B$, then s_g splits all $x \in R$. The reasoning is as before. $\square$

The fact that the proofs of $\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{b} \leq \mathfrak{r}$ resemble each other a lot is no coincidence, as these two proofs are dual to each other. We will talk more about this in Section 2.2.

Using the technique of forcing it is possible to show that none of the inequalities shown so far are provably reversible.

Measure and category

We can define several functions that assign a cardinality to an ideal of the reals. Such functions are called **cardinal functions**. Many cardinal charactacteristics of the continuum can be described as cardinal functions on the reals. We will define four commonly used cardinal functions on ideals:

Definition 2.1.6 — *Cardinal functions on ideals*

Let I be a σ-ideal on $\mathbb{R}$, then we define the following cardinal functions:

- The **uniformity number** $\operatorname{non}(I)$ is the least size of a set $N \subseteq \mathbb{R}$ such that $N \notin I$.
- The **covering number** $\operatorname{cov}(I)$ is the least size of a set $C \subseteq I$ such that $\bigcup C = \mathbb{R}$.
- The **additivity number** $\operatorname{add}(I)$ is the least size of a set $A \subseteq I$ such that $\bigcup A \notin I$.
- The **cofinality number** $\operatorname{cof}(I)$ is the least size of a set $F \subseteq I$ such that every $i \in I$ has an $f \in F$ such that $i \subseteq f$. $\triangleleft$

We will only work with two specific σ-ideals, namely the sets null and meagre, which we defined in the first chapter.

The following theorem states that we can freely choose the perfect Polish space in which we are working, as was promised at the beginning of Section 1.2.

Theorem 2.1.7
If X and Y are nonempty perfect Polish spaces, then all cardinal functions on the ideals null and meagre are identical on X and Y. $\triangleleft$

The following diagram, known as **Cichoń's diagram**, gives an overview of the provable relations between the sizes of the cardinal functions on the ideals null and meagre and the cardinal characteristics $\mathfrak{b}$ and $\mathfrak{d}$. An arrow $X \to Y$ means that $X \geq Y$ is provable. As before, none of the reverse inequalities is provable.

$$\begin{tikzcd}
& \mathrm{cov}(\mathsf{null}) \arrow[dd] & \mathrm{non}(\mathsf{meagre}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathsf{meagre}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathsf{null}) \arrow[l] \arrow[dd] & \mathfrak{c} \arrow[l] \\
& & \mathfrak{b} \arrow[d] & \mathfrak{d} \arrow[l] \arrow[d] & & \\
\aleph_1 & \mathrm{add}(\mathsf{null}) \arrow[l] & \mathrm{add}(\mathsf{meagre}) \arrow[l] & \mathrm{cov}(\mathsf{meagre}) \arrow[l] & \mathrm{non}(\mathsf{null}) \arrow[l] &
\end{tikzcd}$$

For our purposes we will only work with the covering and uniformity numbers on the ideals null and meagre. We will therefore only prove those relations concerning $\mathfrak{b}$, $\mathfrak{d}$ and the aforementioned relevant cardinal functions.

Theorem 2.1.8
$\aleph_1 \leq \mathrm{cov}(\mathsf{null}) \leq \mathrm{non}(\mathsf{meagre}) \leq \mathfrak{c}$ and $\aleph_1 \leq \mathrm{cov}(\mathsf{meagre}) \leq \mathrm{non}(\mathsf{null}) \leq \mathfrak{c}$. $\triangleleft$

Proof. $\mathbb{R}$ has nonzero measure, and thus is not a null set. Since $\mathbb{R}$ is a Polish space it is Baire, so the countable intersection of open dense sets is dense. Any meagre set, being the complement of a countable intersection of open dense sets, cannot contain an interval, which shows $\mathbb{R}$ is not meagre. These two facts show that $\mathfrak{c}$ is an upper bound to both $\mathrm{non}(\mathsf{meagre})$ and $\mathrm{non}(\mathsf{null})$. Furthermore, a countable union of null sets is null by σ-additivity of Lebesgue measure, and the countable union of meagre sets is clearly still meagre, and thus $\mathbb{R}$ cannot be covered by countably many null sets or countably many meagre sets.

We will now prove $\mathrm{cov}(\mathsf{null}) \leq \mathrm{non}(\mathsf{meagre})$ and $\mathrm{cov}(\mathsf{meagre}) \leq \mathrm{non}(\mathsf{null})$. We will represent reals as elements of ${}^{\omega}2$. Define $I_0 = \{0\}$ and recursively define the intervals $I_{n+1} = [\max(I_n)+1, \max(I_n)+n+1]$, then $\{I_n \mid n \in \omega\}$ is an partition of ω. Define the symmetric relation $\approx$ on ${}^{\omega}2$ as $f \approx g$ if $f \upharpoonright I_n = g \upharpoonright I_n$ for infinitely many $n \in \omega$.

Given any $x \in {}^{\omega}2$, let $S_x^n = \{y \in {}^{\omega}2 \mid x \upharpoonright I_n = y \upharpoonright I_n\}$ and let $S_x = \{y \in {}^{\omega}2 \mid x \approx y\}$. Given $k \in \omega$, we can cover S_x with $\bigcup_{n \geq k} S_x^n$. Since $\mu(S_x^n) = 2^{-(n+1)}$ for each n we see that $\bigcup_{n \geq k} S_x^n$ is a cover of S_x with measure 2^{-k}. It follows that $S_x \in \mathsf{null}$.

Let $N_x^k = \{y \in {}^{\omega}2 \mid \forall n > k(x \upharpoonright I_n \neq y \upharpoonright I_n)\}$, then each N_x^k is nowhere dense: if $y \in N_x^k$, and U_s is some basic open around y (i.e. $s = y \upharpoonright m$ for some $m \in \omega$), then take some $n > k$ such that m is smaller than all elements of I_n and extend s to a function $t : \max(I_n) + 1 \to 2$ such that $t \restriction I_n = x \restriction I_n$ to see that $U_t \cap N_x^k = \varnothing$. Therefore $\bigcup_{k\in\omega} N_x^k$ is meagre, and it is the complement of S_x.

Suppose $X \notin \mathsf{null}$, then for any null set N there is some $x \in X$ such that $x \notin N$. Let $y \in {}^\omega 2$, and consider the set $S_y \in \mathsf{null}$, then let $x \in X$ such that $x \notin S_y$. It follows by symmetry of $\approx$ that $y \notin S_x$, and thus $y \in {}^\omega 2 \setminus S_x$. This shows that $\{{}^\omega 2 \setminus S_x \mid x \in X\}$ covers ${}^\omega 2$, and thus there is a cover of ${}^\omega 2$ with $|X|$ meagre sets. This shows that $\mathrm{cov}(\mathsf{meagre}) \leq \mathrm{non}(\mathsf{null})$. The proof of $\mathrm{cov}(\mathsf{null}) \leq \mathrm{non}(\mathsf{meagre})$ is exactly the same, except that the roles of null and meagre sets are reversed throughout the proof. □

Theorem 2.1.9

$\mathfrak{b} \leq \mathrm{non}(\mathsf{meagre})$ and $\mathrm{cov}(\mathsf{meagre}) \leq \mathfrak{d}$. ◁

Proof. We work in ${}^\omega\omega$. Suppose that $B \subseteq {}^\omega\omega$ and $|B| < \mathfrak{b}$, then there is some $f \in {}^\omega\omega$ such that $g \leq^* f$ for all $g \in B$. Let $L_f = \{g \in {}^\omega\omega \mid g \leq^* f\}$. Given $n \in \omega$, let $L_f^n = \{g \in {}^\omega\omega \mid \forall k > n(g(k) \leq f(k))\}$ and take some $g \in L_f^n$. For a basic open U_s containing g, we have $s = g \restriction m$ with m the length of s. Let $m' > m, n$ and extend s to some $t : m' + 1 \to \omega$ such that $t(m') > f(m')$, then $U_t \cap L_f^n = \varnothing$. This shows that L_f^n is nowhere dense for each $n \in \omega$, and thus that $L_f = \bigcup_{n\in\omega} L_f^n$ is meagre. Therefore $B \subseteq L_f$ is meagre as well.

Let $D \subseteq {}^\omega\omega$ be a dominating family, and take some $g \in {}^\omega\omega$. Let $f \in D$ be such that $g \leq^* f$, then $g \in L_f$. Hence $\{L_f \mid f \in D\}$ forms a cover of ${}^\omega\omega$ with meagre sets. □

We can also prove a relation between the covering and uniformity numbers and the splitting and reaping numbers $\mathfrak{s}$ and $\mathfrak{r}$. In particular, $\mathfrak{r}$ is above both covering numbers, and $\mathfrak{s}$ is below both uniformity numbers.

Theorem 2.1.10

$\mathfrak{s} \leq \min\{\mathrm{non}(\mathsf{null}), \mathrm{non}(\mathsf{meagre})\}$ and $\max\{\mathrm{cov}(\mathsf{null}), \mathrm{cov}(\mathsf{meagre})\} \leq \mathfrak{r}$. ◁

Proof. Let $A \subseteq [\omega]^\omega$ be a set of infinite subsets of ω with $|A| < \mathfrak{s}$, then there is some subset $B \in [\omega]^\omega$ such that B is not split by any set in A. We will see that this implies that A is both null and meagre.

Given $f \in {}^\omega 2$, and $n \in \omega$ let $K_f^n = \{g \in {}^\omega 2 \mid \forall m > n(f(m) = 1 \to g(m) = 1)\}$. If f is the characteristic function of an infinite subset of ω, then K_f^n is a null set, because we can fix an arbitrary number of $m_1, \ldots, m_k > n$ for which $f(m_i) = 1$ for each i and take the open set $U(m_1, \ldots, m_k) = \{g \in {}^\omega 2 \mid g(m_1) = 1 \wedge \ldots \wedge g(m_k) = 1\} \supseteq K_f^n$ with measure 2^{-k}. We can also see that K_f^n is nowhere dense, since for any basic open $U_s = \{g \in {}^\omega 2 \mid g \sqsupseteq s\}$ extending some finite initial $s : m \to 2$, we can pick a $k > m, n$ such that $f(k) = 1$ and take the open $V_s^k = \{g \in {}^\omega 2 \mid g \sqsupseteq s \wedge g(k) = 0\}$, then $K_f^n \cap V_s^k$ is empty.

Clearly the same holds for the sets $L_f^n = \{g \in {}^\omega 2 \mid \forall m > n(f(m) = 1 \to g(m) = 0)\}$.

It is easy to see that the set X_A of characteristic functions for the sets in A is a subset of $\bigcup_{n\in\omega} K_f^n \cup L_f^n$ with f the characteristic function for B. This union is a countable union of null nowhere dense sets, and thus X_A is a subset of a null meagre set, making X_A itself null and meagre. We can conclude that any set of cardinality smaller than $\mathfrak{s}$ must be null and meagre.

If $A \subseteq [\omega]^\omega$ is a set witnessing the property of $\mathfrak{r}$, that is, no set $B \in [\omega]^\omega$ splits all the elements of A, then consider the set X_A of characteristic functions for the sets in A. Let $C = \left\{\bigcup_{n\in\omega} K_f^n \cup L_f^n \;\middle|\; f \in X_A\right\}$, then C is a set of null meagre sets. Suppose $B \in [\omega]^\omega$ and let g be its characteristic function. Choose $F \in A$ with characteristic function f such that B does not split F, then $g \in \bigcup_{n\in\omega} K_f^n \cup L_f^n$, since g either shares almost all the 1's of f, or g shares almost no 1's with f. □

2.2 Dual Cardinal Characteristics

We have already hinted that the similarity in the proofs from the last section was no coincidence. In this section we will formalise this idea by introducing **Tukey connections**. Almost all cardinal characteristics we will discuss can be represented as the "norm" of a certain triple of sets. A Tukey connection is a morphism between such triples that implies an order between their norms.

Definition 2.2.1 — *Tukey connections*
Let $\mathscr{X} = \langle X^-, X^+, X\rangle$ be a **relational system** (from now on simply called *triple*) with X^- and X^+ sets and $X \subseteq X^- \times X^+$ a relation. We call X^- the set of **challenges** and X^+ the set of **responses**, where a challlenge $x^- \in X^-$ is **met** by response $x^+ \in X^+$ when $\langle x^-, x^+\rangle \in X$. We define the **norm** $||\mathscr{X}||$ of such a triple $\mathscr{X}$ as the least cardinality of a subset $A \subseteq X^+$ such that every challenge $x \in X^-$ is met by at least one response $a \in A$.

If $\mathscr{X} = \langle X^-, X^+, X\rangle$ is a triple, the **dual** of $\mathscr{X}$ is the triple $\mathscr{X}^\perp = \left\langle X^+, X^-, (X^{-1})^{\mathrm{c}}\right\rangle$, where $\langle x^+, x^-\rangle \in (X^{-1})^{\mathrm{c}}$ if and only if $\langle x^-, x^+\rangle \notin X$.

A **Tukey connection** $\varphi : \mathscr{X} \to \mathscr{Y}$ between triples $\mathscr{X} = \langle X^-, X^+, X\rangle$ and $\mathscr{Y} = \langle Y^-, Y^+, Y\rangle$ is a pair of maps $\varphi = \langle \varphi^-, \varphi^+\rangle$ with $\varphi^- : Y^- \to X^-$ and $\varphi^+ : X^+ \to Y^+$ such that for any $x \in X^+$ and $y \in Y^-$ for which $\langle \varphi^-(y), x\rangle \in X$ we also have $\langle y, \varphi^+(x)\rangle \in Y$. ◁

The usefulness of relational systems to describe cardinal characteristics becomes apparent with the following theorem.

Theorem 2.2.2
If $\varphi : \mathscr{X} \to \mathscr{Y}$ is a Tukey connection, then $||\mathscr{X}|| \geq ||\mathscr{Y}||$ and $\left|\left|\mathscr{Y}^\perp\right|\right| \geq \left|\left|\mathscr{X}^\perp\right|\right|$. ◁

Proof. Let $A \subseteq X^+$ be a set of responses for $\mathscr{X}$ that meets all challenges $x \in X^-$ and let $y \in Y^-$ be a challenge for $\mathscr{Y}$. Since $\varphi^-(y) \in X^-$, there is some $a \in A$ such that $\langle \varphi^-(y), a \rangle \in X$. By the definition of a Tukey connection it then follows that $\langle y, \varphi^+(x) \rangle \in Y$, thus $\varphi^+(x)$ is a response meeting y. It follows that the image $\varphi^+[A]$ is a set of responses for $\mathscr{Y}$ that meets all challenges $y \in Y^-$. Therefore we get that $||\mathscr{X}|| \geq ||\mathscr{Y}||$, because $|\varphi^+[A]| \leq |A|$.

We have $\left|\left|\mathscr{Y}^\perp\right|\right| \geq \left|\left|\mathscr{X}^\perp\right|\right|$, as it is easy to see that $\varphi^\perp = \langle \varphi^+, \varphi^- \rangle : \mathscr{Y}^\perp \to \mathscr{X}^\perp$ is a Tukey connection. □

We will use $\mathscr{X}$ and $\mathscr{Y}$ to denote arbitrary triples. All other times a caligraphic letter is used we will mean a specific fixed triple, such as the ones defined in the following definition and several definitions in later sections. For a quick reference we refer to the table on page 79.

If we look back at the cardinal characteristics from the previous section, we can see they are the norms of triples.

Proposition 2.2.3

Define the following triples:

$$\mathscr{B} = \langle {}^\omega\omega, {}^\omega\omega, \not\geq^* \rangle$$
$$\mathscr{D} = \langle {}^\omega\omega, {}^\omega\omega, \leq^* \rangle$$
$$\mathscr{S} = \langle [\omega]^\omega, [\omega]^\omega, S \rangle \qquad \text{where} \qquad S = \{ \langle a, b \rangle \in [\omega]^\omega \times [\omega]^\omega \mid |a \cap b| = |a \setminus b| = \aleph_0 \}$$
$$\mathscr{R} = \left\langle [\omega]^\omega, [\omega]^\omega, (S^{-1})^{\mathrm{c}} \right\rangle$$
$$\mathscr{C}_I = \langle \mathbb{R}, \mathcal{I}, \in \rangle$$
$$\mathscr{N}_I = \langle \mathcal{I}, \mathbb{R}, \not\ni \rangle$$

We have duals $\mathscr{B}^\perp = \mathscr{D}$; $\mathscr{S}^\perp = \mathscr{R}$ and $\mathscr{C}_\mathcal{I}^\perp = \mathscr{N}_\mathcal{I}$ and norms $||\mathscr{B}|| = \mathfrak{b}$, $||\mathscr{D}|| = \mathfrak{d}$, $||\mathscr{S}|| = \mathfrak{s}$, $||\mathscr{R}|| = \mathfrak{r}$, $||\mathscr{C}_\mathcal{I}|| = \mathrm{cov}(\mathcal{I})$ and $||\mathscr{N}_\mathcal{I}|| = \mathrm{non}(\mathcal{I})$. ◁

Because of Theorem 2.1.7, if $\mathcal{I}$ is equal to null or meagre, we can replace $\mathbb{R}$ with any other perfect Polish space in the above definitions of the triples $\mathscr{C}_\mathcal{I}$ and $\mathscr{N}_\mathcal{I}$.

Many proofs of the previous section can be formulated in terms of Tukey connections. For example, the proofs of $\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{b} \leq \mathfrak{r}$ are the result of the same Tukey connection, as are the proofs of $\mathrm{cov}(\mathsf{null}) \leq \mathrm{non}(\mathsf{meagre})$ and $\mathrm{cov}(\mathsf{meagre}) \leq \mathrm{non}(\mathsf{null})$, and of $\mathfrak{b} \leq \mathrm{non}(\mathsf{meagre})$ and $\mathrm{cov}(\mathsf{meagre}) \leq \mathfrak{d}$.

Example 2.2.4

A Tukey connection $\varphi : \mathscr{D} \to \mathscr{S}$ is obtained by $\varphi^- : [\omega]^\omega \to {}^\omega\omega$ and $\varphi^+ : {}^\omega\omega \to [\omega]^\omega$ where $\varphi^- : x \mapsto f_x$ and $\varphi^+ : g \mapsto s_g$, as defined in the proof of Theorem 2.1.4. The proof goes on to show that φ is a Tukey connection by proving that if $f_x \leq^* g$, then $\langle x, s_g \rangle \in S$, or in words, then s_g splits x. From Theorem 2.2.2 we get both that $||\mathscr{D}|| = \mathfrak{d} \geq \mathfrak{s} = ||\mathscr{S}||$ and that $\left|\left|\mathscr{D}^\perp\right|\right| = \mathfrak{b} \leq \mathfrak{r} = \left|\left|\mathscr{S}^\perp\right|\right|$.

A Tukey connection $\varphi : \mathscr{N}_{\mathsf{meagre}} \to \mathscr{C}_{\mathsf{null}}$ is given by $\varphi^- : {}^\omega 2 \to \mathsf{null}$ and $\varphi^+ : \mathsf{meagre} \to {}^\omega 2$ where $\varphi^- : x \mapsto S_x$ and $\varphi^+ : x \mapsto {}^\omega 2 \setminus S_x$, as defined in the proof of Theorem 2.1.8. The proof shows that if $x \notin S_y$ for some $x, y \in {}^\omega 2$, then $y \in {}^\omega 2 \setminus S_x$, which means that φ is a Tukey connection. Hence $\operatorname{cov}(\mathsf{null}) \leq \operatorname{non}(\mathsf{meagre})$ and $\operatorname{cov}(\mathsf{meagre}) \leq \operatorname{non}(\mathsf{null})$.

A Tukey connection $\varphi : \mathscr{N}_{\mathsf{meagre}} \to \mathscr{B}$ is given by $\varphi^- : {}^\omega\omega \to \mathsf{meagre}$ and $\varphi^+ : {}^\omega\omega \to {}^\omega\omega$ where $\varphi^- : f \mapsto L_f$, as defined in the proof of Theorem 2.1.9, and φ^+ is the identity function. The rest of the proof shows that φ is a Tukey connection by showing that L_f is indeed meagre, which gives that for any $f, g \in {}^\omega\omega$, if $g \notin L_f$, then $g \not\leq^* f$. Hence $\mathfrak{b} \leq \operatorname{non}(\mathsf{meagre})$ and $\operatorname{cov}(\mathsf{meagre}) \leq \mathfrak{d}$. ◁

For some purposes it will be useful to combine multiple triples to be able to say something about both norms at the same time.

Definition 2.2.5 — *Sequential composition of triples*
Let $\mathscr{X} = \langle X^-, X^+, X\rangle$ and $\mathscr{Y} = \langle Y^-, Y^+, Y\rangle$ be two triples. Define their **sequential composition** and **dual sequential composition** as the triples

$$\mathscr{X}^\frown\mathscr{Y} = \left\langle X^- \times {}^{X^+}Y^-, X^+ \times Y^+, \overline{Z}\right\rangle,$$
$$\mathscr{X}_\smile\mathscr{Y} = \left\langle X^- \times Y^-, X^+ \times {}^{X^-}Y^+, \underline{Z}\right\rangle.$$

Here ${}^{X^+}Y^-$ is the set of functions from X^+ to Y^- and similar for ${}^{X^-}Y^+$. We define $\overline{Z}$ as $(x^-, f)\ \overline{Z}\ (x^+, y^+)$ if and only if $x^-\ X\ x^+$ **and** $f(x^+)\ Y\ y^+$, and $\underline{Z}$ as $(x^-, y^-)\ \underline{Z}\ (x^+, f)$ if and only if $x^-\ X\ x^+$ **or** $y^-\ Y\ f(x^-)$. These are dual operators by $\mathscr{X}^\frown\mathscr{Y} = (\mathscr{X}^\perp{}_\smile\mathscr{Y}^\perp)^\perp$. ◁

Lemma 2.2.6
Let $||\mathscr{X}||$ and $||\mathscr{Y}||$ be infinite. Then

$$||\mathscr{X}^\frown\mathscr{Y}|| = \max\{||\mathscr{X}||, ||\mathscr{Y}||\} \qquad \text{and} \qquad ||\mathscr{X}_\smile\mathscr{Y}|| = \min\{||\mathscr{X}||, ||\mathscr{Y}||\}.$$ ◁

Proof. Assume without loss of generality that $||\mathscr{X}|| \geq ||\mathscr{Y}||$.

There is an obvious Tukey connections $\varphi : \mathscr{X}^\frown\mathscr{Y} \to \mathscr{X}$ with $\varphi^- : (x^-, f) \mapsto x^-$ and $\varphi^+ : x^- \mapsto (x^-, y^-)$ for some fixed y^-, and an almost as obvious Tukey connection $\psi : \mathscr{X}^\frown\mathscr{Y} \to \mathscr{Y}$ given by $\psi^- : y^- \mapsto (x^-, f_{y^-})$ with f_{y^-} the constant function to y^- and $\psi^+ : (x^+, y^+) \mapsto y^+$.

On the other hand, let $A \subseteq X^+$ and $B \subseteq Y^+$ satisfy the properties of $\mathscr{X}$ and $\mathscr{Y}$, that is, let A and B be sets of responses that meet all challenges of their respective triple. Then the set $A \times B$ satisfies the property of $\mathscr{X}^\frown\mathscr{Y}$.

The case for dual sequential composition is similar. □

Definition 2.2.7 — *Union of triples*
Let $\mathscr{X} = \langle X, Y, R\rangle$ and $\mathscr{Y} = \langle X, Y, S\rangle$ be two triples with the same set of challenges and responses. Define their **union** as the triple $\mathscr{X} \cup \mathscr{Y} = \langle X, Y, R \cup S\rangle$ and their **intersection** as the triple $\mathscr{X} \cap \mathscr{Y} = \langle X, Y, R \cap S\rangle$. ◁

Lemma 2.2.8
$||\mathscr{X} \cup \mathscr{Y}|| \leq \min\{||\mathscr{X}||, ||\mathscr{Y}||\}$ and $||\mathscr{X} \cap \mathscr{Y}|| \geq \max\{||\mathscr{X}||, ||\mathscr{Y}||\}$. ◁

Proof. The Tukey connections $\mathscr{X} \to \mathscr{X} \cup \mathscr{Y}$ and $\mathscr{X} \cap \mathscr{Y} \to \mathscr{X}$ with all maps equal to the identity map are as required. □

Note that either inequality may be strict. For example if $\mathscr{X} = \langle \mathbb{R}, \mathbb{R}, \leq \rangle$ and $\mathscr{Y} = \langle \mathbb{R}, \mathbb{R}, \geq \rangle$, then $||\mathscr{X}|| = ||\mathscr{Y}|| = \aleph_0$, but $||\mathscr{X} \cup \mathscr{Y}|| = 1$ and $||\mathscr{X} \cap \mathscr{Y}|| = 2^{\aleph_0}$.

2.3 Equivalent Triples

When two triples have the same norm, we call them **equivalent**. For example, the triples $\langle \mathbb{R}, \mathsf{null}, \in \rangle$, $\langle {}^{\omega}2, \mathsf{null}, \in \rangle$ and $\langle {}^{\omega}\omega, \mathsf{null}, \in \rangle$ are all equivalent because $\mathrm{cov}(\mathsf{null})$ is independent of the perfect Polish space that is being used.

It is important to note that equivalence of two triples does not imply that their duals are also equivalent, as is seen by the following example.

Example 2.3.1
Let us say that $f \in {}^{\omega}\omega$ **strongly dominates** $g \in {}^{\omega}\omega$ or g is **strongly bounded** by f, denoted $f \geq g$ when $f(n) \geq g(n)$ for all $n \in \omega$. A **strongly dominating family** X is a set such that any function in ${}^{\omega}\omega$ is strongly dominated by an element of X, and similarly we can define a **strongly unbounded family**.

Let X be a dominating family of cardinality $\mathfrak{d}$, and let $Y = \bigcup X_n$ with

$$X_n = \{f \in {}^{\omega}\omega \mid \exists g \in X \forall k \in \omega (f(k) = g(k) + n)\}.$$

Then Y is also a dominating family, and as Y is a countable union of sets of cardinality $\mathfrak{d}$, we see that $|Y| = \mathfrak{d}$ as well. However, Y is also a strongly dominating family: for any $f \in {}^{\omega}\omega$ there is some $g \in X$ such that $f \leq^* g$. Let N be the maximum of the values $f(n)$ on which $f(n) > g(n)$, which exist since there are only finitely many of such n. Then $g' : n \mapsto g(n) + N$ strongly dominates f and $g' \in X_N \subseteq Y$.

On the other hand, the set $Z = \{f \in {}^{\omega}\omega \mid \exists k \in \omega \forall n \in \omega (f(n) = k)\}$ of constant functions is strongly unbounded.

From this we can conclude that the triples $\mathscr{D} = \langle {}^{\omega}\omega, {}^{\omega}\omega, \leq^* \rangle$ and $\mathscr{D}' = \langle {}^{\omega}\omega, {}^{\omega}\omega, \leq \rangle$ have the same norm $\mathfrak{d}$, but $\mathscr{D}^{\perp} = \langle {}^{\omega}\omega, {}^{\omega}\omega, \not\geq^* \rangle$ has norm $\mathfrak{b}$, while $\mathscr{D}'^{\perp} = \langle {}^{\omega}\omega, {}^{\omega}\omega, \not\geq \rangle$ has norm $\aleph_0$. ◁

This example also shows that if we have two norms $||\mathscr{A}|| \geq ||\mathscr{B}||$, then a Tukey connection $\mathscr{A} \to \mathscr{B}$ need not exist. If we can find Tukey connections $\mathscr{A} \to \mathscr{B} \to \mathscr{A}$, however, we can conclude that $||\mathscr{A}|| = ||\mathscr{B}||$ and $\left|\left|\mathscr{A}^{\perp}\right|\right| = \left|\left|\mathscr{B}^{\perp}\right|\right|$.

We will use this to formulate a few alternative triples that are equivalent to the ones given in Proposition 2.2.3, since in later proofs it will often be easier to work with such alternative.

Proposition 2.3.2
The following triples are equivalent and have equivalent duals:

$$\mathscr{B} = \langle {}^\omega\omega, {}^\omega\omega, \ngeq^* \rangle$$
$$\underline{\mathscr{B}} = \langle \uparrow({}^\omega\omega), \uparrow({}^\omega\omega), \underline{B} \rangle \quad \text{where}$$
$$\underline{B} = \{\langle f, g\rangle \mid \exists^\infty n(|\mathrm{ran}(g) \cap [f(n), f(n+1))| \leq 1)\}$$
$$\mathscr{J} = \langle [\omega]^\omega, \mathcal{S}(\omega), J \rangle \quad \text{where}$$
$$J = \{\langle X, \pi\rangle \mid \exists^\infty x, y \in X(x < y \wedge \pi(x) > \pi(y))\}$$ ◁

Here $\mathscr{B}$ is the triple with norm $\mathfrak{b}$ that was defined before.

The triple $\underline{\mathscr{B}}$ can be best understood as an **interval partition** on ω. We can retrieve the intervals from $f \in \uparrow({}^\omega\omega)$ by considering the intervals $[f(n), f(n+1))$ for each $n \in \omega$ (as well as the interval $[0, f(0))$ if $f(0) \neq 0$). The relation $\underline{B}$ says that if $\langle f, g\rangle \in \underline{B}$, then there are infinitely many n such that the interval $[f(n), f(n+1))$ contains at most a single value in the range of g. This is equivalent to saying that there are infinitely many $n \in \omega$ such that the interval $[f(n), f(n+1))$ does not contain any closed interval $[g(k), g(k+1)]$ with $k \in \omega$.

Finally the triple $\mathscr{J}$ has the **jumbling number** $\mathfrak{j}$ as norm. If $\langle X, \pi\rangle \in J$ for an infinite subset $X \in [\omega]^\omega$ and a permutation π, then we say that X is **jumbled** by π. We have that π jumbles X when there are infinitely many pairs $x, y \in X$ such that their order is reversed by π.

We will give Tukey connections $\mathscr{B} \to \underline{\mathscr{B}} \to \mathscr{J} \to \mathscr{B}$, showing that each of their norms is equal to $\mathfrak{b}$, and dually the norms of their duals to $\mathfrak{d}$. We start with $\mathscr{B} \to \underline{\mathscr{B}}$.

Proposition 2.3.3
There is a Tukey connection $\varphi : \mathscr{B} \to \underline{\mathscr{B}}$. ◁

Proof. Define $\varphi^- : \uparrow({}^\omega\omega) \to {}^\omega\omega$ to be the function such that for $m \in [f(n), f(n+1))$ we have $\varphi^-(f) : m \mapsto f(n+2) - 1$. Define $\varphi^+ : {}^\omega\omega \to \uparrow({}^\omega\omega)$ to be the map that sends g to a function g' that we define recursively such that $g'(0) = 0$ and for any $n \in \omega$ we have $g'(n+1) = \max\{g(0), g(1), \ldots, g(g'(n)), g'(n)\} + 1$.

Suppose that $f \in \uparrow({}^\omega\omega)$ and $g \in {}^\omega\omega$, let $f' := \varphi^-(f)$ and $g' = \varphi^+(g)$. Then we have to show that if $f' \ngeq^* g$, then $\langle f, g'\rangle \in \underline{B}$. So let $\langle x_n \mid n \in \omega\rangle \in \uparrow({}^\omega\omega)$ be a sequence such that $f'(x_n) < g(x_n)$ for all $n \in \omega$. Since g' is strictly increasing, we can find for each x_n some $k_n \in \omega$ such that $g(x_n) \in [g'(k_n), g'(k_n+1))$.

Now x_n lies in some interval $[f(m), f(m+1))$, and thus $f'(x_n) = f(m+2) - 1$. Therefore we see that:

$$f(m+1) \leq f(m+2) - 1 = f'(x_n) < g(x_n) < g'(k_n+1).$$

We will see that the interval $[f(m+1), f(m+2))$ contains at most one value in the range of g'. If $g'(k_n) \leq f(m+1)$ this is immediate. If $g'(k_n) > f(m+1)$, suppose that $g'(k_n - 1) \geq f(m+1)$, then for any $y \leq g'(k_n - 1)$ we see that $g(y) < g'(k_n)$ by how we defined g'. But since $g'(k_n - 1) \geq f(m+1) > x_n$, we see that $g'(k_n) > g(x_n)$, which contradicts $g(x_n) \in [g'(k_n), g'(k_n+1))$. Therefore we can see that $g'(k_n-1) < f(m+1) < f(m+2) < g'(k_n+1)$, and thus $[f(m+1), f(m+2))$ contains at most one value in the range of g'. $\square$

Proposition 2.3.4
There is a Tukey connection $\varphi : \underline{\mathscr{B}} \to \mathscr{J}$. $\triangleleft$

Proof. For a set $X \in [\omega]^\omega$, let $f_X : \omega \to X$ be the (unique) order-preserving bijection. Now define $\varphi^- : [\omega]^\omega \to \uparrow({}^\omega\omega)$ as the function that sends $X \in [\omega]^\omega$ to the function $f : n \mapsto f_X(3n)$. Let $\varphi^+ : \uparrow({}^\omega\omega) \to \mathcal{S}(\omega)$ send a function $g \in \uparrow({}^\omega\omega)$ to a permutation π such that if $x \in [g(n), g(n+1))$, then $\pi(x) = g(n) + g(n+1) - x - 1$.

Suppose that $X \in [\omega]^\omega$ and $g \in \uparrow({}^\omega\omega)$ and let $f = \varphi^-(X)$ and $\pi = \varphi^+(g)$. Then we have to show that if $\langle f, g\rangle \in \underline{B}$, then X is jumbled by π. Let $h = f_X$ be the order-preserving bijection between ω and X, and let $\langle k_n \mid n \in \omega\rangle$ be an infinite sequence witnessing that $\langle f, g\rangle \in \underline{B}$. Then there is no more than a single value in the range of g contained in each interval $I_n = [f(k_n), f(k_n+1)) = [h(3k_n), h(3k_n+3))$.

Let $m \in \omega$ such that $g(m) \leq h(3k_n+2) < g(m+1)$. Since one of $g(m) \notin I_n$ or $g(m+1) \notin I_n$ must hold, we see that the interval $[g(m), g(m+1))$ contains one of $h(3k_n+1)$ or $h(3k_n+3)$. But in the case it contains $h(3k_n+1)$ we then see by h being order-preserving that:

$$\begin{aligned}\pi(h(3k_n+2)) &= g(m) + g(m+1) - h(3k_n+2) - 1\\ &< g(m) + g(m+1) - h(3k_n+1) - 1 = \pi(h(3k_n+1)).\end{aligned}$$

And in case it contains $h(3k_n+3)$ we see:

$$\begin{aligned}\pi(h(3k_n+2)) &= g(m) + g(m+1) - h(3k_n+2) - 1\\ &> g(m) + g(m+1) - h(3k_n+3) - 1 = \pi(h(3k_n+3)).\end{aligned}$$

This implies that π will jumble X, since the order of at least two of the three elements $h(3k_n+1)$, $h(3k_n+2)$ and $h(3k_n+3)$ of X is reversed by π for infinitely many k_n. $\square$

Proposition 2.3.5
There is a Tukey connection $\varphi : \mathscr{J} \to \mathscr{B}$. $\triangleleft$

Proof. For $g \in {}^\omega\omega$, let $g' \in \uparrow({}^\omega\omega)$ be the least strictly increasing function larger than g (with the order as usual on ${}^\omega\omega$), then define $\varphi^- : {}^\omega\omega \to [\omega]^\omega$ to be the function that sends g to the set $X = \{x_i \mid i \in \omega\}$ such that $x_0 = 0$ and $x_{i+1} = g'(x_i)$ for all $i \in \omega$.

Let the function $\varphi^+ : \mathcal{S}(\omega) \to {}^\omega\omega$ send $\varphi^+(\pi)$ to the function f that sends $n \in \omega$ to $\max(\pi^{-1}[\max(\pi[n+1])] + 1) + 1$, that is, we take maximum m of $\pi(0), \ldots, \pi(n)$, then take the maximum k of $\pi^{-1}(0), \ldots, \pi^{-1}(m)$, and let $f(n) = k + 1$. It follows that for any $x \leq n$ and $y \geq f(n)$ we have $\pi(x) < \pi(y)$.

Suppose that $g \in {}^\omega\omega$ and $\pi \in \mathcal{S}(\omega)$, and let $X = \varphi^-(g)$ and $f = \varphi^+(\pi)$. Then we have to show that if X is jumbled by π, then $g \not\geq^* f$. Suppose that $x_i, x_j \in X$ are elements such that $x_i < x_j$ and $\pi(x_j) < \pi(x_i)$, then we know from how we defined f that $x_j < f(x_i)$. But we also have $g(x_i) \leq g'(x_i) = x_{i+1} \leq x_j$, and therefore $g(x_i) < f(x_i)$. If X is jumbled by π, there are infinitely many x_i with some x_j such that $\pi(x_j) < \pi(x_i)$, and thus there are infinitely many points x_i where $g(x_i) < f(x_i)$, showing that $g \not\geq^* f$. $\square$

We will give one more characterisation of the bounding number, which expresses the same idea as $\underline{\mathscr{B}}$, but using infinite coinfinite sets of natural numbers instead.

Proposition 2.3.6

The following triples are equivalent and their duals are equivalent:

$$\underline{\mathscr{B}} = \langle {\uparrow}({}^\omega\omega), {\uparrow}({}^\omega\omega), \underline{B} \rangle$$
$$\overline{\mathscr{B}} = \left\langle [\omega]^\omega_\omega, [\omega]^\omega_\omega, \overline{B} \right\rangle \quad \text{where}$$
$$\overline{B} = \{\langle X, Y\rangle \mid \exists^\infty x, y \in X (x < y \wedge Y \cap [x, y) = \varnothing)\} \qquad \triangleleft$$

Proof. We will give Tukey connections $\underline{\mathscr{B}} \xrightarrow{\varphi} \overline{\mathscr{B}} \xrightarrow{\psi} \underline{\mathscr{B}}$.

For a set $X \in [\omega]^\omega_\omega$, let $g' : \omega \to X$ be the unique order isomorphism and define $\varphi^-(X)$ to be the function $g : n \mapsto g'(2n)$. Given a function $f \in {\uparrow}({}^\omega\omega)$ let $\varphi^+(f)$ be the set $Y = \{f(2n) \mid n \in \omega\}$. If $\langle g, f\rangle \in \underline{B}$, then we can find arbitrarily large $n \in \omega$ such that $[g(n), g(n+1)) = [g'(2n), g'(2n+2))$ contains only a single value in $\operatorname{ran}(f)$. Since $g'(2n), g'(2n+1), g'(2n+2) \in X$ we have that either $[g'(2n), g'(2n+1))$ or $[g'(2n+1), g'(2n+2))$ does not intersect $Y \subseteq \operatorname{ran}(f)$, thus $\langle X, Y\rangle \in \overline{B}$.

On the other hand, for $f \in {\uparrow}({}^\omega\omega)$ define $\psi^-(f)$ to be the set $Y = \{f(2n) \mid n \in \omega\}$ and for $X \in [\omega]^\omega_\omega$ define $\psi^+(X)$ to be the unique order isomorphism $g' : \omega \to X$. If $\langle Y, X\rangle \in \overline{B}$, then we can find arbitrarily large $x, y \in X$ such that $x < y$ and $[x, y)$ does not intersect Y. Since $X = \operatorname{ran}(g')$ we have some $n, m \in \omega$ such that $[g'(n), g'(m)) \cap Y = \varnothing$. Without loss of generality we can take $m = n + 1$, then by definition of Y we can find $k \in \omega$ such that:

$$f(2k) < g'(n) < f(2k+1) \leq g'(n+1) \leq f(2k+2).$$

$[g'(n), g'(n+1))$ contains at most a single value in the range of f. $\square$

Corollary 2.3.7

$$||\mathscr{B}|| = ||\underline{\mathscr{B}}|| = \left|\left|\overline{\mathscr{B}}\right|\right| = ||\mathscr{J}|| = \mathfrak{b} \text{ and } ||\mathscr{D}|| = \left|\left|\underline{\mathscr{B}}^{\perp}\right|\right| = \left|\left|\overline{\mathscr{B}}^{\perp}\right|\right| = \left|\left|\mathscr{J}^{\perp}\right|\right| = \mathfrak{d} \qquad \triangleleft$$

We will also give another triple $\underline{\mathscr{S}}$ that is equivalent to the triple $\mathscr{S}$, but uses coinfinite subsets of ω instead.

Proposition 2.3.8

The following triples are equivalent and their duals are equivalent:

$$\mathscr{S} = \langle [\omega]^\omega, [\omega]^\omega, S \rangle$$
$$\underline{\mathscr{S}} = \langle [\omega]^\omega_\omega, [\omega]^\omega_\omega, S \cap ([\omega]^\omega_\omega \times [\omega]^\omega_\omega) \rangle \qquad \triangleleft$$

Proof. There is an obvious Tukey connection $\varphi : \underline{\mathscr{S}} \to \mathscr{S}$, with φ^- any map that restricts to the identity on coinfinite sets and φ^+ the identity. Let $X \in [\omega]^\omega$ and $Y \in [\omega]^\omega_\omega$ such that $\varphi^-(X)$ is split by Y. If X is coinfinite, then $X = \varphi^-(X)$ is split by $\varphi^+(Y) = Y$ as well. On the other hand, if X is cofinite, then Y splits X automatically since any cofinite set is split by an infinite coinfinite set.

A Tukey connection $\psi : \mathscr{S} \to \underline{\mathscr{S}}$ is not much more difficult. We let ψ^- be the identity and ψ^+ be any map that restricts to the identity on coinfinite sets. If $X \in [\omega]^\omega_\omega$ and $Y \in [\omega]^\omega$ and $\varphi^-(X) = X$ is split by Y, then $X \setminus Y$ is infinite, thus Y cannot be cofinite. It follows that X is split by $\psi^+(Y) = Y$ as well. □

2.4 Forcing Strict Inequalities

It is a well-known fact that any of the relations drawn in Cichoń's diagram can consistently be strict. Even stronger yet, as long as the equalities $\min\{\mathfrak{b}, \mathrm{cov}(\mathsf{meagre})\} = \mathrm{add}(\mathsf{meagre})$ and $\max\{\mathfrak{d}, \mathrm{non}(\mathsf{meagre})\} = \mathrm{cof}(\mathsf{meagre})$ are preserved, any assignment of the cardinalities $\aleph_1$ and $\aleph_2 = \mathfrak{c}$ to the cardinal characteristics faithful to the diagram can be shown to be consistent. Such consistency proofs employ the method of forcing.

We will use this section to give an overview of how the cardinal characteristics relevant to our exposition will behave in some models of ZFC. The four forcings will give us six models that are enough to show that any strict relation that does not contradict the results of the previous sections is consistent. We will not give full details of the proofs, as this will cost too much space. We will try to give some motivation for each claim, however.

Cohen forcing

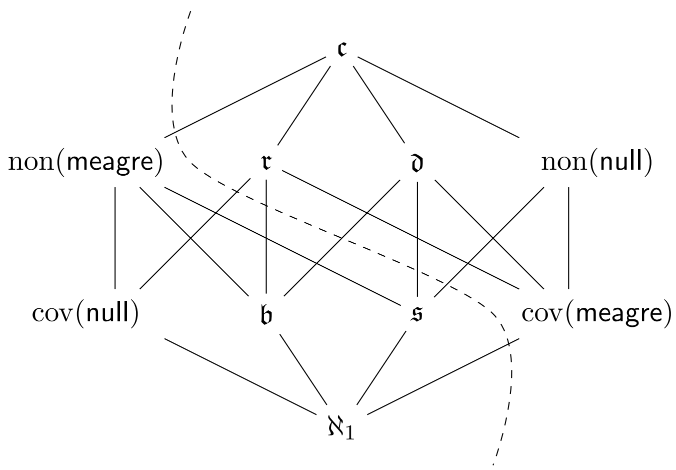
We will discuss the size of the cardinal characteristics in the Cohen model. Remember that the Cohen model is the model $\mathcal{M}[G]$, with G being generic for $\mathbb{C}(\aleph_2)$ over a ctm $\mathcal{M} \vDash \mathsf{GCH}$.

For the Cohen model we already have all the tools we need to determine the size of the relevant cardinals. This is because, as we saw in Theorem 1.3.24, a Cohen real falls outside any meagre set coded in the ground model. This implies that cov(meagre) has size $\aleph_2$ in the Cohen model, as we saw in Example 1.3.32.

On the other hand, we can see that non(meagre) $= \aleph_1$ in the Cohen model. This is because every set of $\aleph_1$ Cohen reals is nonmeagre. Let $\{r_\alpha \mid \alpha < \kappa\}$ be the set of Cohen reals that are added by $\mathbb{C}(\kappa)$, and let $A = \{\alpha_\xi \mid \xi < \omega_1\} \subseteq \kappa$ be a set of indicies. Let M be a meagre set in $\mathcal{M}[G]$, then $M \subseteq M'$ where M' is a meagre set coded by some real r. Since we can see $\mathbb{C}(\kappa)$ as an iteration, we can find some $\beta \in \kappa$ such that $r \in \mathcal{M}[G_\beta]$.

Note that with Cohen forcing we could permute the order of the iteration without changing the resulting model. We saw in Example 1.3.32 that the way to view Cohen forcing as an iteration is by viewing a finite partial function $f : \kappa \nrightarrow 2$ as a condition p of the iteration $\mathbb{P}_\kappa$ by letting $p(\alpha)(n) = f(\alpha + n)$ for all $\alpha + n \in \mathrm{dom}(f)$. We can permute the order of iteration with some permutation $\pi : \kappa \to \kappa$ by first mapping all conditions $f : \kappa \nrightarrow 2$ to conditions $g : \kappa \nrightarrow 2$ where $g(\alpha + n) = f(\pi(\alpha) + n)$, and then sending g to a condition of the iteration P_κ. Because of this, we can assume that the real r that defines the meagre set M' is added in some countable stage of the iteration. Since $|A| = \aleph_1$, we know that there is some $\xi < \omega_1$ for which r_{α_ξ} is added in a stage of the iteration after r, and thus this $r_{\alpha_\xi} \notin M'$. This shows that $\left\{r_{\alpha_\xi} \mid \xi < \omega_1\right\}$ is not contained in any meagre set, and thus is nonmeagre.[1]

Since cov(null), $\mathfrak{b}, \mathfrak{s} \leq$ non(meagre), and non(null), $\mathfrak{d}, \mathfrak{r} \geq$ cov(meagre), we see that the two results from above completely determine the size of the cardinal characteristics that we are interested in. The following Hasse diagram depicts the Cohen model, where the dotted line separates the cardinals of size $\aleph_1$ from the cardinals of size $\mathfrak{c}$:



Random forcing

The random model is the model $\mathcal{M}[G]$, where G is a generic filter for the random forcing $\mathbb{B}(\aleph_2)$ over a ctm $\mathcal{M} \models$ GCH. As with Cohen forcing, we can decide the cardinality of most of the relevant cardinal characteristics by looking at the null ideal.

[1]We do need a set of ω_1 Cohen reals, since any countable set of Cohen reals can be defined from a single Cohen real, because an iteration of countable length of $\mathbb{C}(\aleph_0)$ is countable, and thus equivalent to $\mathbb{C}(\aleph_0)$ itself.

For very similar reasons to the situation in Cohen forcing, we have that $\mathrm{cov}(\mathsf{null}) = \aleph_2$ and $\mathrm{non}(\mathsf{null}) = \aleph_1$ after forcing with $\mathbb{B}(\aleph_2)$. The argument is a little different, since we cannot see $\mathbb{B}(\kappa)$ as a finite support iteration (because finite support iterations add Cohen reals, and $\mathbb{B}(\kappa)$ does not add a Cohen real). It is possible to view $\mathbb{B}(\kappa)$ as an iteration with a different support, and, like Cohen forcing, it is also allowed to permute the order in which the random reals are added. From this it will follow that $\mathrm{cov}(\mathsf{null}) = \aleph_2$ and $\mathrm{non}(\mathsf{null}) = \aleph_1$ by the same argument we saw with Cohen forcing.

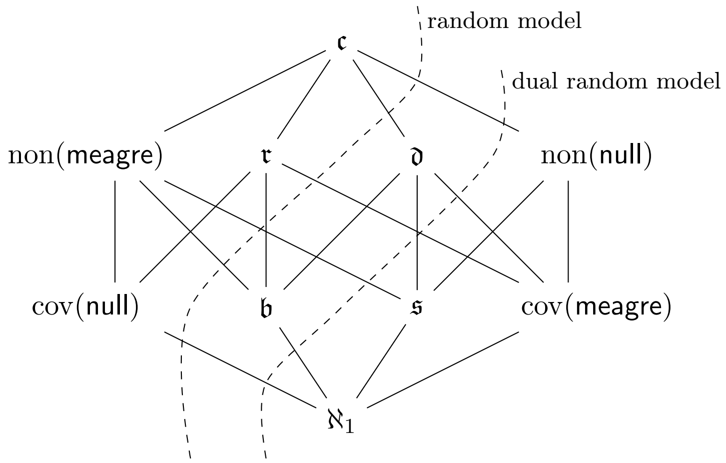
Since $\mathrm{non}(\mathsf{meagre})$ and $\mathfrak{r}$ are larger than $\mathrm{cov}(\mathsf{null})$ and $\mathrm{cov}(\mathsf{meagre})$ and $\mathfrak{s}$ are smaller than $\mathrm{non}(\mathsf{null})$, it follows that except for $\mathfrak{b}$ and $\mathfrak{d}$ the cardinality of all relevant cardinal characteristics is determined in the random model.

The effect that random forcing has on $\mathfrak{b}$ and $\mathfrak{d}$ is that their cardinality stays equal to their cardinality in the ground model. This is because random forcing is ${}^{\omega}\omega$-bounding: any real $f \in {}^{\omega}\omega \cap \mathcal{M}[G]$ is dominated by some $g \in {}^{\omega}\omega \cap \mathcal{M}$. To see this, let $\dot{f}$ be a name for f and let $C \in \mathcal{B}({}^{\kappa}2)$ be a compact set of positive measure $\mu(C)$. For every $n \in \omega$ we can find some subset $C_{x_n} \subseteq C$ of measure $\mu(C_{x_n}) = (1 - 2^{-n-2}) \cdot \mu(C)$ such that $C_{x_n} \Vdash \dot{f}(n) < x_n$. The compact set $C' = \bigcap_{n\in\omega} C_{x_n}$ then has positive measure $\mu(C') \geq \frac{1}{2}\mu(C)$, and thus it is a condition. Furthermore, if we let $g \in {}^{\omega}\omega$ be the function with $g(n) = x_n$ for all $n \in \omega$, then we see that $C' \Vdash \forall n \in \omega(\dot{f}(n) \leq g(n))$. Since $\mathbb{B}(\kappa)$ is ccc, we can find a countable set $\{g_k \mid k \in \omega\}$ of such functions g such that $C \Vdash \exists k \in \omega \forall n \in \omega(\dot{f}(n) \leq g_k(n))$. Since the set $\{g_k \mid k \in \omega\}$ is countable, we can find a function h that dominates all g_k, and thus $C \Vdash \dot{f} \leq^* h$.

This means that in the random model, we have $\mathfrak{b} = \mathfrak{d} = \aleph_1$, because the ground model satisfies GCH. But this also implies that we could start with a different model where $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \aleph_2$ and force with $\mathbb{B}(\aleph_2)$. This will have the effect that $\mathfrak{b}$ and $\mathfrak{d}$ stay of cardinality $\aleph_2$, while $\mathrm{non}(\mathsf{null})$ becomes $\aleph_1$ and $\mathrm{cov}(\mathsf{null})$ becomes $\aleph_2$.

An example of a model where $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \aleph_2$ is a model of $\mathsf{ZFC} + \mathfrak{c} = \aleph_2 + \mathsf{MA}$. Here MA, known as **Martin's axiom**, is the statement that for every ccc forcing poset $\mathbb{P}$ with $|\mathbb{P}| < \mathfrak{c}$ there exists a generic filter G. The theory $\mathsf{ZFC} + \mathfrak{c} = \aleph_2 + \mathsf{MA}$ is consistent and can be forced from a model of ZFC by doing a specific finite support iterated forcing of length $\aleph_2$. A consequence of MA is that any cardinal characteristic that can be forced to become large using a ccc forcing with cardinality $< \mathfrak{c}$ is large in a model of MA. In particular all the cardinal characteristics from the Cichoń diagram are of cardinality $\mathfrak{c}$ in a model of MA.

We will therefore define the **dual random model** to be the result of forcing with $\mathbb{B}(\aleph_2)$ over a ctm $\mathcal{M} \vDash \mathfrak{c} = \aleph_2 + \mathsf{MA}$. We have drawn the situation in both the random model and the dual random model in the following Hasse diagram:



Hechler forcing

In order to show the consistency of $\operatorname{cov}(\mathsf{null}) < \mathfrak{b}$ and $\operatorname{non}(\mathsf{null}) > \mathfrak{d}$ we can use finite support iteration of the following forcing.

Definition 2.4.1 — *Hechler forcing*
The **Hechler forcing** $\mathbb{D}$ has as conditions $(s, F) \in {}^{<\omega}\omega \times [{}^{\omega}\omega]^{<\omega}$ and as ordering $(t, G) \leq (s, F)$ if $t \supseteq s$, $G \supseteq F$ and $t(n) \geq f(n)$ for all $f \in F$ and $n \in \operatorname{dom}(t) \setminus \operatorname{dom}(s)$. $\triangleleft$

Hechler forcing is ccc. In fact it is σ-centred, since the family of conditions (s, F) for a fixed $s \in {}^{<\omega}\omega$ form a centred set.

Let $\mathcal{M} \vDash \mathsf{GCH}$ be a ctm, and let $\mathbb{D}_{\omega_2}$ be an ω_2-length finite support iteration of Hechler forcing, and let G be a generic filter for $\mathbb{D}_{\omega_2}$, then $\mathcal{M}[G]$ is called the **Hechler model**. We will have $\mathcal{M}[G] \vDash \aleph_1 = \operatorname{cov}(\mathsf{null}) = \mathfrak{s} < \mathfrak{b} = \operatorname{cov}(\mathsf{meagre}) = \aleph_2$.

The reason that $\operatorname{cov}(\mathsf{null})$ is small, is a consequence of Hechler forcing being σ-centered and thus not adding random reals by Theorem 1.3.30. However, we do not force with the poset $\mathbb{D}$, but with a finite support iteration of $\mathbb{D}$, and even though $\mathbb{D}$ is σ-centred, this does not imply that the iteration $\mathbb{D}_{\omega_2}$ is σ-centred. For this we will need the following preservation theorem:

Theorem 2.4.2
If $\mathbb{P}_\gamma = \left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \;\middle|\; \alpha < \gamma \right\rangle$ is a finite support iteration of σ-centred forcings, then forcing with $\mathbb{P}_\gamma$ does not add random reals. $\triangleleft$

To see that $\mathfrak{b}$ is large, we define the notion of a **dominating real** being a real $f \in {}^{\omega}\omega \in \mathcal{M}[G]$ such that $f \geq^* g$ for every $g \in {}^{\omega}\omega \cap \mathcal{M}$. Forcing with $\mathbb{D}$ adds a dominating real, since the set $\bigcup\{s \mid (s, F) \in G\}$ dominates all functions from the ground model. This is easy to see by a density argument, since for any (s, F) with $\operatorname{dom}(s) = n$, $g \in {}^{\omega}\omega$ and $m > n$ we can extend (s, F) to $(t, F \cup \{g\})$ such that $m \in \operatorname{dom}(t)$ and $t(m) > g(m)$. Because we have an iteration of length ω_2, any set of $\aleph_1$ many reals in $\mathcal{M}[G]$ is present in some $\mathcal{M}[G_\alpha]$, and thus there is a real in $\mathcal{M}[G_{\alpha+1}]$ that dominates all of the $\aleph_1$ reals. It follows that $\mathcal{M}[G] \vDash \mathfrak{b} = \aleph_2$.

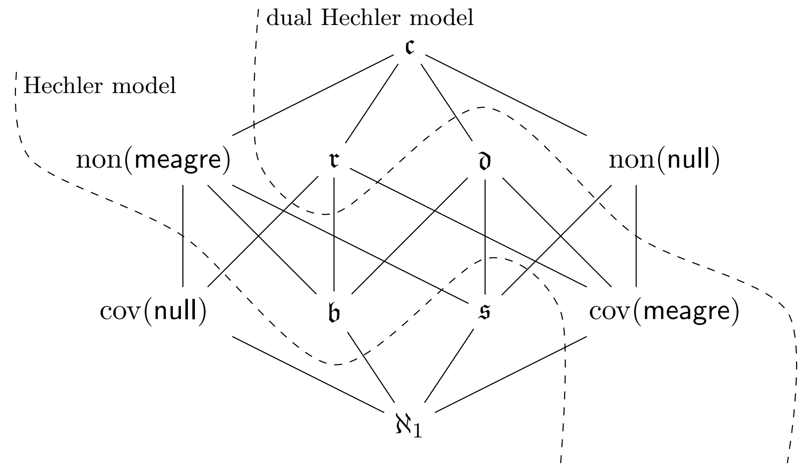
Finally cov(meagre) is large because we use a finite support iteration of length ω_2, thus we add $\aleph_2$ many Cohen reals. Alternatively this follows from the fact that the forcing $\mathbb{D}$ itself also adds a Cohen real at each step.

Similarly to what we did with random forcing, if we let $\mathcal{N} \vDash \mathsf{MA} + \mathfrak{c} = \aleph_2$ be a ctm and let $\mathbb{D}_{\omega_1}$ be a ω_1-length finite support iteration of Hechler forcing with H a generic filter for $\mathbb{D}_{\omega_1}$, then $\mathcal{N}[H]$ is called the **dual Hechler model**. We get $\mathcal{N}[H] \vDash \aleph_1 = \text{non}(\mathsf{meagre}) = \mathfrak{d} < \text{non}(\mathsf{null}) = \mathfrak{r} = \aleph_2$.

This time $\mathfrak{d}$ is small because we add only $\aleph_1$ many dominating functions, non(meagre) is small since we add $\aleph_1$ many Cohen reals that together form a nonmeagre set and non(N) stays large because no random reals are added.

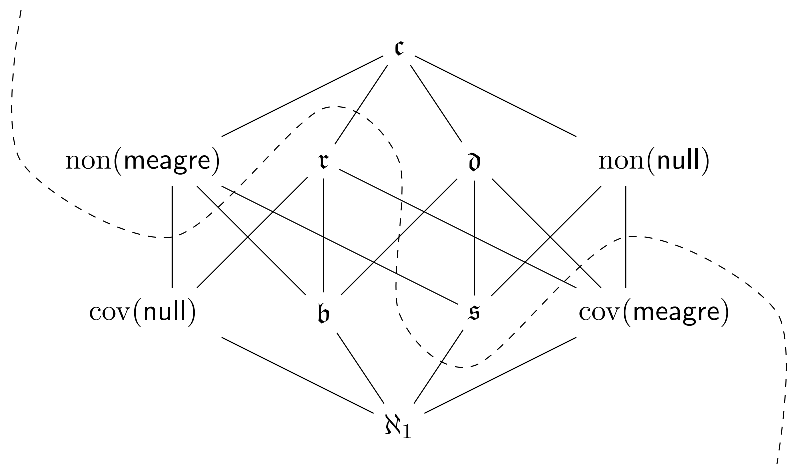
For a proof that $\mathcal{M}[G] \vDash \mathfrak{s} = \aleph_1$ and $\mathcal{N}[H] \vDash \mathfrak{r} = \mathfrak{c}$ we refer to Theorem 2.8 in Minami [24].

We can draw the Hechler and dual Hechler models in a Hasse diagram as follows:



Blass-Shelah forcing

The last model we will mention is the **Blass-Shelah model**, which useful to us since it will have $\mathfrak{r} < \mathfrak{s}$. We will not go into any details about this model, since it is a countable support iteration of a non-ccc forcing. Instead we refer to Blass and Shelah [5] for more information.



Chapter 3

The Rearrangement Numbers

The rearrangement numbers are a small family of cardinal characteristics that arise from combinatorial questions about the Riemann Rearrangement Theorem. They were first studied by the authors of [4] as the result of a question by Michael Hardy on MathOverflow [15].

3.1 Conditionally Convergent Series

In this section we will introduce the rearrangement theorem, due to Riemann [30].

Definition 3.1.1 — *Infinite series*
Let $\langle a_n \mid n \in \omega \rangle \in {}^{\omega}\mathbb{R}$ be an infinite sequence of real numbers. We will use the shorthand $\overline{a}$ to denote $\langle a_n \mid n \in \omega \rangle$. Remember that an infinite series is either **convergent** to some limit in $\mathbb{R}$, **divergent** to infinity (positive or negative) or divergent by **oscillation**.

A point $p \in [-\infty, \infty]$ in the extended real line is called an **accumulation point** of $\overline{a}$ if there is an increasing sequence $\langle n_i \mid i \in \omega \rangle$ such that the sequence of partial sums $\left\langle \sum_{k<n_i} a_n \;\middle|\; i \in \omega \right\rangle$ converges to p if $p \in \mathbb{R}$ or diverges to infinity if $p \in \{-\infty, \infty\}$.

A convergent series is **absolutely convergent** if the sum of its absolute terms $\sum_{n\in\omega} |a_n|$ is convergent, and **conditionally convergent** if it is not absolutely convergent. Note that a convergent series is conditionally convergent if and only if both the sum of its positive terms and the sum of its negative terms diverge to infinity. We will say a **sequence** $\overline{a} \in {}^{\omega}\mathbb{R}$ is conditinally convergent if the associated series $\sum_{n\in\omega} a_n$ is conditionally convergent. The set of all conditionally convergent sequences is denoted as $\textsc{ccs} \subseteq {}^{\omega}\mathbb{R}$. ◁

Theorem 3.1.2 — *Riemann Rearrangement Theorem*
If $\overline{a}$ is a conditionally convergent sequence, then there exist ...

- ... a permutation $\pi_x \in \mathcal{S}(\omega)$ for any $x \in \mathbb{R}$ such that $\sum_{n\in\omega} a_{\pi(n)} = x$,
- ... permutations $\sigma^+, \sigma^- \in \mathcal{S}(\omega)$ such that $\sum_{n\in\omega} a_{\sigma^+(n)} = \infty$ and $\sum_{n\in\omega} a_{\sigma^-(n)} = -\infty$,
- ... a permutation $\tau \in \mathcal{S}(\omega)$ such that $\sum_{n\in\omega} a_{\tau(n)}$ diverges by oscillation. ◁

In the early twentieth century this theorem has been generalised by Lévy [21] and Steinitz [34] to infinite series of vectors in $\mathbb{R}^n$. Before we can mention the theorem, we first need an alternative for *conditionally convergent* that works for series of vectors. We will use boldface to emphasise that we mean vectors in $\mathbb{R}^n$ instead of reals in $\mathbb{R}$.

Definition 3.1.3 — *Independence*
Let $\overline{\boldsymbol{a}} : \omega \to \mathbb{R}^n$ be an infinite sequence of vectors in $\mathbb{R}^n$. Let $K(\overline{\boldsymbol{a}}) \subseteq \mathbb{R}^n$ be the subspace consisting of those $\boldsymbol{s} = \langle s^1, \dots, s^n \rangle \in \mathbb{R}^n$ such that $\sum_{k\in\omega} \boldsymbol{s} \cdot \boldsymbol{a}_k$ is absolutely convergent (here $\boldsymbol{s} \cdot \boldsymbol{a}_k$ is the dot product $s^1 a_k^1 + \cdots + s^n a_k^n$). Define $R(\overline{\boldsymbol{a}})$ as the orthogonal complement of $K(\overline{\boldsymbol{a}})$.

A sequence $\overline{\boldsymbol{a}} : \omega \to \mathbb{R}^n$ is **indepedent** if $R(\overline{\boldsymbol{a}}) = \mathbb{R}^n$. A set $I \subseteq {}^\omega\mathbb{R}$ of infinite series in $\mathbb{R}$ is called **independent** if the infinite sequence $\langle \langle a_k^1, \dots, a_k^n \rangle \mid k \in \omega \rangle$ in $\mathbb{R}^n$ is independent for every finite set $\{\overline{a}^1, \dots, \overline{a}^n\} \subseteq I$ and any $n \in \omega$. ◁

Theorem 3.1.4 — *Lévy-Steinitz Theorem*
If $\overline{\boldsymbol{a}} : \omega \to \mathbb{R}^n$ is an infinite sequence of vectors such that $\sum_{k\in\omega} \boldsymbol{a}_k$ converges to a vector $\boldsymbol{t} \in \mathbb{R}^n$, and $\boldsymbol{s} \in R(\overline{\boldsymbol{a}})$, then there is a permutation $\pi \in \mathcal{S}(\omega)$ such that $\sum_{k\in\omega} \boldsymbol{a}_{\pi(k)} = \boldsymbol{t} + \boldsymbol{s}$. ◁

In particular, if $\overline{\boldsymbol{a}}$ is independent, we see that for any $\boldsymbol{s} \in \mathbb{R}^n$ there is a permutation π such that $\sum_{k\in\omega} \boldsymbol{a}_{\pi(k)} = \boldsymbol{s}$. Steinitz' proof of the theorem uses the following theorem as a lemma, which will also be of use in our application of the theorem in this chapter.

Theorem 3.1.5 — *Polygonal Confinement Theorem*
For each $n \in \omega$ there exists a constant $C_n \in \mathbb{R}$ such that for any $k \in \omega$ and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in \mathbb{R}^n$ with $M = \max\{||\boldsymbol{v}_i|| \mid 1 \le i \le k\}$ and $\sum_{1\le i\le k} \boldsymbol{v}_i = 0$, there exists a permutation π on $\{2, \dots, k\}$ such that $\left|\left|\boldsymbol{v}_1 + \sum_{2\le i\le m} \boldsymbol{v}_{\pi(i)}\right|\right| \le M \cdot C_n$ for any $m \le k$. ◁

What this theorem states, is that we could rearrange the vectors that make up a closed polygonal path starting at the origin with each vector of length less than or equal to M, such that the rearranged polygonal path is completely contained in a ball of radius $M \cdot C_n$ around the origin. Here C_n only depends on the dimension of the space we work in, and not on the number of vectors in the path.

For a modern exposition of the proof of the Lévy-Steinitz Theorem and the Polygonal Confinement Theorem, see Rosenthal [31].

3.2 The Rearrangement Numbers

We can define several interesting cardinal characteristics, named the **rearrangement numbers**, based on the result from Riemann's Rearrangement Theorem. They all arise from the question what the least cardinality must be of a set of permutations on ω such that every conditionally convergent sequence $\overline{a}$ will not converge to its original value when permuted with one of the permutations in the set.

Definition 3.2.1 — *Rearrangement number*
Let the **rearrangement number** $\mathfrak{rr}$ be the least cardinality of a set $A \subseteq \mathcal{S}(\omega)$ of permutations such that for every conditionally convergent sequence $\overline{a}$ there is a permutation $\pi \in A$ for which the series $\sum_{n\in\omega} a_{\pi(n)}$ does not converge to the limit of $\sum_{n\in\omega} a_n$. $\triangleleft$

We can define more specific cardinal characteristics by looking at the way in which a permutation makes the conditionally convergent series not converge to the same limit. There are three distinctions: the permuted series could diverge to infinity, it could diverge by oscillation or it could converge to a different value.

Definition 3.2.2 — *Rearrangement numbers*
Let $\mathfrak{rr}_o$, $\mathfrak{rr}_i$ and $\mathfrak{rr}_f$ be the least cardinalities of sets $A_o, A_i, A_f \subseteq \mathcal{S}(\omega)$ respectively such that for every conditionally convergent sequence $\overline{a}$ there is a permutation $\pi_o \in A_o$, $\pi_i \in A_i$ and $\pi_f \in A_f$ for which the series $\sum_{n\in\omega} a_{\pi_o(n)}$ diverges by oscillation, $\sum_{n\in\omega} a_{\pi_i(n)}$ diverges to infinity and $\sum_{n\in\omega} a_{\pi_f(n)}$ converges to a different limit than $\sum_{n\in\omega} a_n$. $\triangleleft$

Finally we could combine these finer distinctions of not converging to the original value to create intermediate cardinal characteristics $\mathfrak{rr}_{io}$, $\mathfrak{rr}_{fi}$ and $\mathfrak{rr}_{fo}$.

The definition of these cardinal characteristics are of a form that is suitable to describe them as the norm of a triple. We have the following triples for the characteristics $\mathfrak{rr}_i$, $\mathfrak{rr}_o$ and $\mathfrak{rr}_f$:

Definition 3.2.3 — *Triples for the rearrangement numbers*
Remember that CCS is the set of conditional convergent sequences and $\mathcal{S}(\omega)$ is the set of permutations on ω. We define the following three triples:

$$\mathscr{R}_o = \langle \text{CCS}, \mathcal{S}(\omega), R_o \rangle, \quad \mathscr{R}_i = \langle \text{CCS}, \mathcal{S}(\omega), R_i \rangle, \quad \mathscr{R}_f = \langle \text{CCS}, \mathcal{S}(\omega), R_f \rangle \quad \text{where}$$

$$\begin{aligned}
R_o &= \left\{ \langle \overline{a}, \pi \rangle \mid \textstyle\sum_\omega a_{\pi(n)} \text{ diverges by oscillation} \right\} \\
R_i &= \left\{ \langle \overline{a}, \pi \rangle \mid \textstyle\sum_\omega a_{\pi(n)} \text{ diverges to } +\infty \text{ or } -\infty \right\} \\
R_f &= \left\{ \langle \overline{a}, \pi \rangle \mid \textstyle\sum_\omega a_{\pi(n)} \text{ converges and } \textstyle\sum_\omega a_n \neq \textstyle\sum_\omega a_{\pi(n)} \right\}
\end{aligned}$$

Then $||\mathscr{R}_o|| = \mathfrak{rr}_o$, $||\mathscr{R}_i|| = \mathfrak{rr}_i$ and $||\mathscr{R}_f|| = \mathfrak{rr}_f$. $\triangleleft$

We can get triples for the other four rearrangement numbers by taking unions, as defined in Definition 2.2.7. By doing so we see that we can define the other four rearrangement numbers as norms of the following triples:

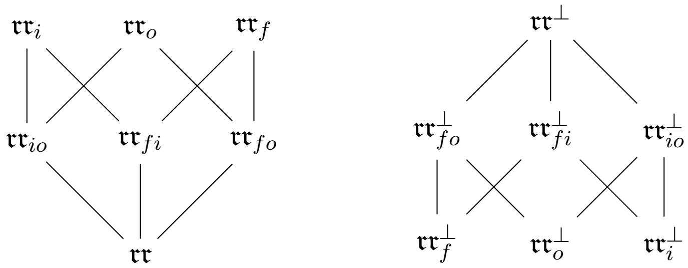
$$\begin{aligned}
\mathscr{R}_{io} &= \mathscr{R}_i \cup \mathscr{R}_o & ||\mathscr{R}_{io}|| &= \mathfrak{rr}_{io} \\
\mathscr{R}_{fi} &= \mathscr{R}_f \cup \mathscr{R}_i & ||\mathscr{R}_{fi}|| &= \mathfrak{rr}_{fi} \\
\mathscr{R}_{fo} &= \mathscr{R}_f \cup \mathscr{R}_o & ||\mathscr{R}_{fo}|| &= \mathfrak{rr}_{fo} \\
\mathscr{R}_{fio} &= \mathscr{R}_f \cup \mathscr{R}_i \cup \mathscr{R}_o & ||\mathscr{R}_{fio}|| &= \mathfrak{rr}
\end{aligned}$$

Because the rearrangement numbers can be formulated as norms of triples, we have a range of dual cardinal characteristics as well. We will denote the duals in the same way we denote duals of triples, for example, $\mathfrak{rr}^\perp$ is the dual of $\mathfrak{rr}$.

Definition 3.2.4 — *Dual rearrangement numbers*
The **dual rearrangement number $\mathfrak{rr}^\perp$** is the least cardinality of a set $C \subseteq \textsc{ccs}$ of conditionally convergent sequences such that there is no permutation $\pi \in \mathcal{S}(\omega)$ for which for all $\bar{a} \in C$ we have $\sum_{n\in\omega} a_{\pi(n)} \neq \sum_{n\in\omega} a_n$. ◁

That is, $\mathfrak{rr}^\perp$ is the least size of a set of conditionally convergent series such that no single permutation simultaneously makes all of the permuted sums different than the original. Similarly, $\mathfrak{rr}_i^\perp$ is the least cardinality of a set of conditionally convergent sequences such no permutation makes all of them diverge to infinity, and similar for the other refinements of the rearrangement numbers.

The Tukey connection described in Lemma 2.2.8 shows us that the rearrangement numbers and their duals are ordered as depicted in the following Hasse diagram. The ordering is also clear from inspection of their definitions.



3.3 Rearrangement Numbers and other Cardinal Characteristics

There are several upper and lower bounds provable in ZFC that can be given to the rearrangement numbers in terms of other cardinal characteristics. In this section we will discuss these bounds, in particular the relation between the rearrangement numbers themselves, the bounding and dominating numbers and the covering and uniformity numbers over the null and meagre ideals.

All of the results pertaining to the rearrangement numbers are based on the results from the paper [4]. The relevance of what we will do in this thesis is that we will reshape the proofs, where possible, into Tukey connections between triples to find out if a dual statement holds for the dual rearrangement numbers.

One very interesting result from [4] is that not all rearrangement numbers are distinct from each other. In the paper it is shown that $\mathfrak{rr} = \mathfrak{rr}_o$. Unfortunately the proof from the paper does not readily translate to a Tukey connection. This means that we will have to do some additional work if we wish to show that the same holds true in the dual case.

First, we will discuss why the proof from [4] does not work for the dual case. The key idea of the proof is that we can convert a permutation for which a permuted series converges to a different limit or diverges to infinity into a new permutation that will make the permuted series oscillate. This conversion can be described as a Tukey connection between $\mathscr{R}_{fi}$ and $\mathscr{R}_o$.

The essential idea is to mix two permutations with two different accumulation points into a new permutation that has both points as accumulation points.

Definition 3.3.1 — *Mixing permutations*
Let $\pi, \sigma \in \mathcal{S}(\omega)$ be permutations. We will call $\tau \in \mathcal{S}(\omega)$ a **mixing** of π and σ if $\tau[n] = \pi[n]$ for infinitely many n and $\tau[n] = \sigma[n]$ for infinitely many n. In this text we will only mix permutations with the identity, and thus we will restrict our case to constructing a mixing between π and the identity.

We will construct a mixing permutation τ recursively. Let $n_0 = 0$, and let τ_0 be the empty bijection on $\varnothing = 0$, then $\tau_0[0] = 0$. Suppose that τ_{2k} is defined and is a bijection on $[0, n_{2k})$. Let $M = \pi^{-1}[n_{2k}+1]$ be the preimage of π over the interval $[0, n_{2k}]$ and define $n_{2k+1} = \max(M)+1$, then we see that $n_{2k+1} > n_{2k}$ as well as $[0, n_{2k}) \subseteq \pi[n_{2k+1}]$. Define τ_{2k+1} to be a bijection between $[0, n_{2k+1})$ and $\pi[n_{2k+1}]$ such that $\tau_{2k+1} \restriction n_{2k} = \tau_{2k}$.

Next, let $K = \pi[n_{2k+1} + 1]$ be the image of π over $[0, n_{2k+1}]$ and define $n_{2k+2} = \max(K) + 1$, then we see that $n_{2k+2} > n_{2k+1}$ as well as $\pi[n_{2k+1}] \subseteq [0, n_{2k+2})$. Define τ_{2k+2} to be a bijection on $[0, n_{2k+2})$ such that $\tau_{2k+2} \restriction [0, n_{2k+1}) = \tau_{2k+1}$.

Finally define $\tau = \bigcup_{k\in\omega} \tau_k$, then τ is a mixing of π and the identity. We will call the sequence $\overline{n} = \langle n_i \mid i \in \omega \rangle$ the **mixing characteristic** of τ. We see that $\overline{n}$ is strictly increasing with $\tau[n_i] = n_i$ for every even $i \in \omega$ and $\tau[n_i] = \pi[n_i]$ for every odd $i \in \omega$. ◁

Theorem 3.3.2
There is a Tukey connection $\varphi : \mathscr{R}_{fi} \to \mathscr{R}_o$. ◁

Proof. Given $\pi \in \mathcal{S}(\omega)$, let $\varphi^+(\pi) = \tau$ be a permutation mixing π with the identity with mixing characteristic $\overline{n}$. We let φ^- be the identity function.

Suppose that $(\overline{a}, \pi) \in R_{fi} \subseteq \textsc{ccs} \times \mathcal{S}(\omega)$. We will show that $\sum_{n\in\omega} a_{\tau(n)}$ diverges by oscillation. For any even $i \in \omega$ we have $\tau[n_i] = n_i$, hence $\sum_{k<n_i} a_{\tau(k)} = \sum_{k<n_i} a_k$. Therefore there is a subseries of $\sum_{n\in\omega} a_{\tau(n)}$ converging to the same limit as $\sum_{n\in\omega} a_n$ does. On the other hand, for any odd $i \in \omega$ we have $\tau[n_i] = \pi[n_i]$, and therefore

$\sum_{k<n_i} a_{\tau(k)} = \sum_{k<n_i} a_{\pi(k)}$, hence there also is a subseries converging to a different limit or diverging towards infinity. This means that $\sum_{n\in\omega} a_{\tau(n)}$ diverges by oscillation. $\square$

The theorem that $\mathfrak{rr} = \mathfrak{rr}_o$ now follows, as we can take a set $A \subseteq \mathcal{S}(\omega)$ of permutations that witnesses the property of $\mathfrak{rr}$ and consider instead the set of the same cardinality $A \cup \varphi^+[A]$, with φ^+ from the above proof. This set will witness the property of $\mathfrak{rr}_o$, since for any $\overline{a} \in$ CCS, we can take $\pi \in A$ such that $\sum_{n\in\omega} a_{\pi(n)} \neq \sum_{n\in\omega} a_n$. If $\sum_{n\in\omega} a_{\pi(n)}$ already oscillates, we have found a suitable permutation in $A \cup \varphi^+[A]$, and otherwise we can pick $\varphi^+(\pi) \in A \cup \varphi^+[A]$ to get a permutation that makes the series of $\overline{a}$ oscillate.

Suppose we want to use this argument to show that $\mathfrak{rr}_o^\perp = \mathfrak{rr}^\perp$ as well. This comes down to showing that if a set $C \subseteq$ CCS has $|C| < \mathfrak{rr}^\perp$, then also $|C| < \mathfrak{rr}_o^\perp$. That is, if there exists a permutation π such that $\sum_{n\in\omega} a_{\pi(n)} \neq \sum_{n\in\omega} a_n$ for all $\overline{a} \in C$, then there exists a permutation π' such that $\sum_{n\in\omega} a_{\pi'(n)}$ diverges by oscillation for all $\overline{a} \in C$.

Using the $\pi' = \varphi^+(\pi)$ from the proof of Theorem 3.3.2 is not sufficient: although π' will make any $\overline{a} \in C$ diverge by oscillation for which $\sum_{n\in\omega} a_{\pi(n)}$ diverges to infinity or converges to a different limit, it will not necessarily do so if $\sum_{n\in\omega} a_{\pi(n)}$ is already divergent by oscillation.

Example 3.3.3
For example, the following sequence

$$\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} \ + \ \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{3} \ + \ \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} \ + \ \cdots$$

is conditionally convergent, but can be rearranged with a permutation π that only swaps some of the negative terms with some of the positive terms to get to the oscillating sequence

$$\tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2} \ + \ \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{3} - \tfrac{1}{3} \ + \ \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} \ + \ \cdots$$

However, $\pi[n] = n$ already holds for infinitely many n, so we could take $\varphi^+(\pi)$ to be the identity permutation. Therefore $\pi' = \varphi^+(\pi)$ might make a sequence that oscillates under permutation by π no longer oscillate under permutation of π'. We will need a different approach to merge the two permutations π (which works for all oscillating series in C) and π' (which works for all other series in C) into a suitable permutation that works for all of C simultaneously. $\triangleleft$

The trick to create a suitable permutation is to use another known fact from the paper, that $\mathfrak{rr} \geq \mathfrak{b}$. The proof from the paper can be dualised by use of a Tukey connection to show that also $\mathfrak{rr}^\perp \leq \mathfrak{d}$. The significance of this result, is that if a set $C \subseteq$ CCS satisfies the property of $\mathfrak{rr}_o^\perp$, but has cardinality $|C| < \mathfrak{rr}^\perp$, then we can use that $|C| < \mathfrak{b}$ as well in our search for a suitable mixing.

We use that there is a rapidly growing function that is not bounded by any functions in a set of functions of cardinality $|C|$. This function can serve as a very rapidly growing mixing characteristic to define a mixing of some permutation π with the identity. This mixing characteristic grows fast enough to show that some accumulation point of the permuted series unequal to the original limit will remain an accumulation point after permuting the series with the newly defined mixing permutation.

We first will dualise the proof that $\mathfrak{rr} \geq \mathfrak{b}$.

Theorem 3.3.4
$\mathfrak{rr} \geq \mathfrak{b}$ and $\mathfrak{rr}^{\perp} \leq \mathfrak{d}$ ◁

Proof. We will give a Tukey connection $\varphi : \mathscr{R}_{fio} \to \mathscr{J}$, where $\mathscr{J}$ is the triple from Proposition 2.3.2. The existence of φ implies that $\mathfrak{rr} \geq \mathfrak{b}$ and $\mathfrak{rr}^{\perp} \leq \mathfrak{d}$.

Define $\varphi^{-} : [\omega]^{\omega} \to \text{CCS}$ as the map that sends X to the sequence $\overline{a}$ where

$$a_n = \begin{cases} 0 & \text{if } n \notin X, \\ \frac{1}{k} & \text{if } n \in X \text{ and } k = |X \cap (n+1)| \text{ is odd}, \\ -\frac{1}{k} & \text{if } n \in X \text{ and } k = |X \cap (n+1)| \text{ is even}. \end{cases}$$

Let $\varphi^{+} : \mathcal{S}(\omega) \to \mathcal{S}(\omega)$ be the identity.

We have to show that for any $X \in [\omega]^{\omega}$ and $\pi \in \mathcal{S}(\omega)$, if π makes the sum corresponding to $\varphi^{-}(X)$ diverge to a different value, then X is jumbled by π. By contraposition if X is not jumbled by π, then at most finitely elements of X are permuted by π, but then in the series only finitely many nonzero terms are permuted, hence the sum of the permuted series does not differ from the original. □

As a direct corollary we see that $\max\{\mathfrak{rr}, \mathfrak{b}\} = \mathfrak{rr}$. Consequently, we can show that $\mathfrak{rr}_o = \mathfrak{rr}$ by showing that $\max\{\mathfrak{rr}, \mathfrak{b}\} \geq \mathfrak{rr}_o$. We will use the sequential composition $\mathscr{R}_{fio}{}^\frown\mathscr{B}$, which has norm $\max\{\mathfrak{rr}, \mathfrak{b}\}$ by Lemma 2.2.6.

Theorem 3.3.5
There is a Tukey connection $\mathscr{R}_{fio}{}^\frown\mathscr{B} \to \mathscr{R}_o$, thus $\max\{\mathfrak{rr}, \mathfrak{b}\} \geq \mathfrak{rr}_o$ and $\min\left\{\mathfrak{rr}^{\perp}, \mathfrak{d}\right\} \leq \mathfrak{rr}_o^{\perp}$. ◁

Proof. The sequential composition $\mathscr{R}_{fio}{}^\frown\mathscr{B}$ is the triple

$$\left\langle \text{CCS} \times {}^{\mathcal{S}(\omega)}({}^{\omega}\omega),\ \mathcal{S}(\omega) \times {}^{\omega}\omega,\ Z \right\rangle,$$

The relation Z is defined in the following way: suppose we have $\overline{a} \in \text{CCS}$, function $f_{(\cdot)} : \mathcal{S}(\omega) \to {}^{\omega}\omega$ sending a permutation σ to $f_{\sigma} \in {}^{\omega}\omega$, permutation $\pi \in \mathcal{S}(\omega)$ and $g \in {}^{\omega}\omega$, then $(\overline{a}, f_{(\cdot)})\ Z\ (\pi, g)$ if and only if $\sum_{n\in\omega} a_{\pi(n)} \neq \sum_{n\in\omega} a_n$ and $f_{\pi} \not\geq^{*} g$.

A Tukey connection $\varphi : \mathscr{R}_{fio}{}^\frown\mathscr{B} \to \mathscr{R}_o$ consists of the functions

$$\varphi^{-} : \text{CCS} \to \text{CCS} \times {}^{\mathcal{S}(\omega)}({}^{\omega}\omega)$$
$$\varphi^{+} : \mathcal{S}(\omega) \times {}^{\omega}\omega \to \mathcal{S}(\omega)$$

We will define φ^- first. Let $\overline{a} \in \text{CCS}$, then $\varphi^- : \overline{a} \mapsto (\overline{a}, f_{(\cdot)})$, with $f_{(\cdot)} : \pi \mapsto f_\pi$ for any permutation π, defined as follows. Let $A \subseteq \mathbb{R} \cup \{-\infty, \infty\}$ be the set of accumulation points of $\sum_{n\in\omega} a_{\pi(n)}$. If $|A| = 1$, let $b \in A$ be the only accumulation point of $\sum_{n\in\omega} a_{\pi(n)}$, and otherwise choose some $b \in A$ such that b is unequal to the limit of the convergent series $\sum_{n\in\omega} a_n$. Let σ be the mixing of π with the identity given in Definition 3.3.1, with mixing characteristic $\overline{n}$.

We will say that $s \in \mathbb{R}$ is n-**close** to b for some $n \in \omega$ if either b is a finite value in $\mathbb{R}$ and $s \in (b - \frac{1}{n}, b + \frac{1}{n})$, or $b = -\infty$ and $s < -n$, or $b = \infty$ and $s > n$.

Given some $n \in \omega$, we define two natural numbers $k_1 > k_0 > n$ such that the partial sums $\sum_{k \le k_i} a_{\pi(k)}$ are n-close to b for both i and such that there is some odd $j \in \omega$ for which $k_0 < n_j < n_{j+1} < k_1$. We now define $f_\pi(n)$ to be this value k_1.

Next we will define φ^+. Let $\pi \in \mathcal{S}(\omega)$ and $g \in {}^\omega\omega$. Once again, let σ be the mixing of π with the identity given in Definition 3.3.1, with mixing characteristic $\overline{n}$. Let $\overline{g} \in {}^\omega\omega$ be the least strictly increasing function such that $g \le \overline{g}$ and such that $\text{ran}(\overline{g}) \subseteq \{n_i \mid i \in \omega \text{ and } i \text{ is odd}\}$. We define $\varphi^+(\pi, g) = \tau$ to be the following recursively defined permutation.

Let $\tau_0 = \pi \restriction n_1$. Since 1 is odd, we see that τ_0 in a bijection between $[0, n_1)$ and $\pi[n_1] = \sigma[n_1]$. Suppose we have constructed τ_i for some $i \in \omega$ such that τ_i is a bijection between $[0, n_j)$ and $\pi[n_j] = \sigma[n_j]$ for some odd j. Let $\overline{g}(n_j) = n_k$, then $k > j$, so we can define τ_{i+1} with domain $[0, n_{k+2})$ as:

- $\tau_{i+1} \restriction [0, n_j) = \tau_i$,
- $\tau_{i+1} \restriction [n_j, n_k) = \pi \restriction [n_j, n_k)$ and
- $\tau_{i+1} \restriction [n_k, n_{k+2}) = \sigma \restriction [n_k, n_{k+2})$

Since j, k and $k+2$ are all odd and $k+1$ is even, we see that

- $\tau_{i+1}[n_j] = \pi[n_j] = \sigma[n_j]$,
- $\tau_{i+1}[n_k] = \pi[n_k] = \sigma[n_k]$,
- $\tau_{i+1}[n_{k+1}] = \sigma[n_{k+1}] = n_{k+1}$ and
- $\tau_{i+1}[n_{k+2}] = \pi[n_{k+2}] = \sigma[n_{k+2}]$.

Finally define $\tau = \bigcup_{i\in\omega} \tau_i$.

With φ^- and φ^+ defined, we are ready to show that φ is a Tukey connection. Let $\overline{a} \in \text{CCS}$, $\pi \in \mathcal{S}(\omega)$ and $g \in {}^\omega\omega$ and define $(\overline{a}, f_{(\cdot)}) = \varphi^-(\overline{a})$ and $\tau = \varphi^+(\pi, g)$. Assume that $(\overline{a}, f_{(\cdot)})\ Z\ (\pi, g)$, then $\sum_{n\in\omega} a_{\pi(n)} \ne \sum_{n\in\omega} a_n$ and $f_\pi \not\ge^* g$. We have to show that $\sum_{n\in\omega} a_{\tau(n)}$ diverges by oscillation.

It is easy to see that the limit of $\sum_{n\in\omega} a_n$ is an accumulation point of $\sum_{n\in\omega} a_{\tau(n)}$, since we have infinitely many $n \in \omega$ such that $\tau[n] = n$. This follows from the assertion $\tau_{i+1}[n_{k+1}] = \sigma[n_{k+1}] = n_{k+1}$ in the construction of τ.

Since $\sum_{n\in\omega} a_{\pi(n)} \neq \sum_{n\in\omega} a_n$, we know that the accumulation point b that was chosen in the construction of φ^- is unequal to $\sum_{n\in\omega} a_n$. We will show that there are arbitrarily large $m \in \omega$ such that for some $k > m$ the k-th partial sum of $\sum_{n\in\omega} a_{\tau(n)}$ is m-close to b. This implies that b is an accumulation point of $\sum_{n\in\omega} a_{\tau(n)}$.

Let $n \in \omega$ be arbitrary, then there exists an $m > n$ such that $f_\pi(m) < g(m)$, since f_π does not dominate g. Remember that we have then the following chain of values:

$$m < k_0 < n_i < n_{i+1} < k_1 = f_\pi(m) < g(m) \leq \overline{g}(m)$$

Assume towards contradiction that $\tau[k_0] \neq \pi[k_0]$ and $\tau[k_1] \neq \pi[k_1]$, then we must have $\tau[k_0] = \sigma[k_0]$ and $\tau[k_1] = \sigma[k_1]$ from the construction of τ. We see that there must be intervals $k_0 \in [n_k, n_{k+2})$ and $k_1 \in [n_{k'}, n_{k'+2})$ for some odd k, k'. We cannot have $k = k'$, as $k_0 < n_{k+2} \leq n_i < n_{i+1} < k_1$. We also know from this construction that $\overline{g}(n_{k+2}) \leq n_{k'}$. But this contradicts that $\overline{g}$ is strictly increasing, as we then have $m < k_0 < n_{k+2} < \overline{g}(n_{k+2}) \leq k_1 < \overline{g}(m)$.

Therefore we can conclude that $\tau[k_i] = \pi[k_i]$ for one of the two i, and thus the partial sum $\sum_{n\leq k_i} a_{\tau(n)}$ is m-close to b. $\square$

Since $\mathfrak{rr}_{io}$ and $\mathfrak{rr}_{fo}$ are in between $\mathfrak{rr}$ and $\mathfrak{rr}_o$, we see that each of $\mathfrak{rr}_o$, $\mathfrak{rr}_{io}$ and $\mathfrak{rr}_{fo}$ are equal to $\mathfrak{rr}$. The same thing holds for their duals. One other result from [4] shows that we can give an even better bound for $\mathfrak{rr}_{fi}$ with respect to the bounding and dominating numbers. This result can be dualised, as we will show in the next theorem.

Theorem 3.3.6
$\mathfrak{rr}_{fi} \geq \mathfrak{d}$ and $\mathfrak{rr}_{fi}^{\perp} \leq \mathfrak{b}$ ◁

Proof. We prove this by giving a Tukey connection $\varphi : \mathscr{R}_{fi} \to \mathscr{D}$, were $\mathscr{D}$ is the triple with norm $\mathfrak{d}$ from Proposition 2.2.3 and we represent the reals as ${}^\omega\omega$.

First we will need some tools to define $\varphi^- : {}^\omega\omega \to \textsc{ccs}$. For a function $g \in {}^\omega\omega$ let $\overline{g}(0) = g(0)$ and $\overline{g} : n \mapsto \max(\overline{g}(n-1)+1, g(n))$ for all $n > 0$. Then let $x_0 = 0$ and $x_{n+1} = \overline{g}(x_n)$ to get an increasing sequence $\langle x_n \mid n \in \omega \rangle$. Now let $\overline{a} \in \textsc{ccs}$, and define $\varphi^-(g)$ to be the sequence $\overline{b}$, with $b_n = a_m$ if $n = x_m$ for some $m \in \omega$, and otherwise $b_n = 0$. We could view this as if we padded $\overline{a}$ with zeroes such that the entries of $\overline{a}$ are now at the locations given by the sequence $\langle x_n \mid n \in \omega \rangle$.

We let $\varphi^+ : \mathcal{S}(\omega) \to {}^\omega\omega$ be the same function we defined in the proof of Proposition 2.3.5.

Suppose that $g \in {}^\omega\omega$ and $\pi \in \mathcal{S}(\omega)$, then let $f = \varphi^+(\pi)$ and let $\overline{b} = \varphi^-(g)$. Let $\langle x_n \rangle$ be the infinite sequence generated from $\overline{g}$ as described above. If $\sum_{n\in\omega} b_{\pi(n)}$ converges to a different limit than $\sum_{n\in\omega} b_n$ or if $\sum_{n\in\omega} b_{\pi(n)}$ diverges to infinity, then we have to show that $g \leq^* f$.

Suppose that y is such that $g(y) > f(y)$. By how we constructed f we have $f(y) > y$, thus also $g(y) > y$. We can find an $n \in \omega$ such that $x_{n+1} > y \geq x_n$, since $\langle x_n \rangle$ is strictly increasing and $x_0 = 0$, then $x_{n+2} = \overline{g}(x_{n+1}) > \overline{g}(y) \geq g(y) > f(y) > y \geq x_n$.

For any $k \leq n$ and $m \geq n+2$ we then have $x_k \leq y$ and $x_m \geq f(y)$, therefore by how we defined f we get $\pi(x_k) < \pi(x_m)$. Now if we take the partial sum of the first x_{n+1} terms of $\bar{b}$, then this is equal to the partial sum of the first $n+1$ terms of $\overline{a}$, as we defined $b_{x_k} = a_k$. Therefore if we take the partial sum of the first $\sup\{\pi(x_k) \mid k \leq n\}$ many terms of $\left\langle b_{\pi(k)} \mid k \in \omega \right\rangle$, we know this contains zero terms, all of the terms $b_{x_k} = a_k$ for $k \leq n$ and possibly the term $b_{x_{n+1}} = a_{n+1}$.

As we know that $\sum_{n\in\omega} b_{\pi(n)}$ does not oscillate or converge to the same limit as $\sum_{n\in\omega} b_n$, we know that there cannot be a sequence of partial sums converging to the limit of $\sum_{n\in\omega} b_n$. Therefore only finitely many y can exist such that $g(y) > f(y)$, as the existence of infinitely many of such y give infinitely many partial sums of $\sum_{n\in\omega} b_{\pi(n)}$ being equal to either the partial sum $\sum_{k\leq n} a_k$ or to the partial sum $\sum_{k\leq n+1} a_k$. □

Cardinal Functions over Ideals on the Reals

In the previous section we found some relations between the rearrangement numbers and the bounding and dominating numbers. In this section we will extend these results by proving some relations between the rearrangement numbers and the cardinal functions $\mathrm{cov}(\mathsf{meagre})$, $\mathrm{non}(\mathsf{meagre})$, $\mathrm{cov}(\mathsf{null})$ and $\mathrm{non}(\mathsf{null})$.

Since $\mathrm{cov}(\mathsf{meagre}) \leq \mathfrak{d}$ and $\mathrm{non}(\mathsf{meagre}) \geq \mathfrak{b}$ we get $\mathfrak{rr}_{fi} \geq \mathrm{cov}(\mathsf{meagre})$ and $\mathfrak{rr}_{fi}^{\perp} \leq \mathrm{non}(\mathsf{meagre})$ as well. We can furthermore use the proof of $\mathrm{non}(\mathsf{meagre}) \geq \mathfrak{rr}$ from [4] to build a Tukey connection, and hence show that $\mathrm{cov}(\mathsf{meagre}) \leq \mathfrak{rr}^{\perp}$.

The proof uses the triple $\mathscr{R}_o$, but as we saw in the last section, this is no problem as $\mathfrak{rr}_o = \mathfrak{rr}$ and $\mathfrak{rr}_o^{\perp} = \mathfrak{rr}^{\perp}$

Theorem 3.3.7
$\mathrm{non}(\mathsf{meagre}) \geq \mathfrak{rr}$ and $\mathrm{cov}(\mathsf{meagre}) \leq \mathfrak{rr}^{\perp}$. ◁

Proof. We will give a Tukey connection $\varphi : \mathscr{N}_{\mathsf{meagre}} \to \mathscr{R}_o$ with the reals represented as $\mathcal{S}(\omega)$ as a subspace of ${}^{\omega}\omega$, that is, $\mathscr{N}_{\mathsf{meagre}} = \langle \mathsf{meagre}, \mathcal{S}(\omega), \in \rangle$.

We let $\varphi^+ : \mathcal{S}(\omega) \to \mathcal{S}(\omega)$ be the identity map and define $\varphi^- : \mathrm{CCS} \to \mathsf{meagre}$ as the function that sends a sequence $\overline{a}$ to the set of permutations $\varphi^-(\overline{a}) = M$ such that $\pi \in M$ if and only if $\sum_{n\in\omega} a_{\pi(n)}$ does not diverge by oscillation. We have to show that M is a meagre set to see that φ^- is a well-defined function.

Let A be the set of permutations π such that $\sum_{n\in\omega} a_{\pi(n)}$ has both partial sums of arbitrarily large positive size and arbitrarily large negative size. Then all permutations in A make $\sum_{n\in\omega} a_{\pi(n)}$ diverge by oscillation, hence $A \cap M = \varnothing$. We can show that A is comeagre, which means that $\mathcal{S}(\omega) \setminus A$ is meagre, and thus $M \subseteq \mathcal{S}(\omega) \setminus A$ as well.

Since A is the intersection of the set B^* of permutations π such that $\sum_{n\in\omega} a_{\pi(n)}$ has partial sums of arbitrarily large positive size and the set B_* of permutations π such that $\sum_{n\in\omega} a_{\pi(n)}$ has partial sums of arbitrarily large negative size, we are finished when we can show B^* and B_* are comeagre, since the intersection of two comeagre sets is comeagre.

Note that we can describe B^* as a countable intersection $B^* = \bigcap_{k\in\omega} U_k$ where

$$U_k = \bigcup_{m\in\omega} \left\{ \pi \in \mathcal{S}(\omega) \;\middle|\; \textstyle\sum_{n\leq m} a_{\pi(n)} \geq k \right\}$$

Each U_k is open, as it is the union of sets of all permutations that extend the finite part $\pi \restriction m$ of some permutation π for which $\sum_{n\leq m} a_{\pi(n)} \geq k$. Moreover, U_k is dense, as for any π, we could take the basic open $V_n = \{\sigma \in \mathcal{S}(\omega) \mid \sigma \restriction n = \pi \restriction n\}$, then $V_n \cap U_k$ is nonempty because we could extend $\pi \restriction n$ to a permutation that permutes enough positive terms to the beginning of the permuted series to make a partial sum become larger than k. So B^* is the countable intersection of open dense sets, and thus B^* is comeagre. Similarly we can show that B_* is comeagre.

The maps φ^- and φ^+ form a Tukey connection, which is now trivial to check: let $\overline{a} \in \textsc{ccs}$ and $\pi \in \mathcal{S}(\omega)$, and define $M = \varphi^-(\overline{a})$. If $\pi \notin M$, then by definition of φ^- we get that $\sum_{n\in\omega} a_{\pi(n)}$ diverges by oscillation. $\square$

In [4], it is proved that $\mathrm{cov}(\mathsf{null}) \leq \mathfrak{rr}$. The proof can be translated in terms of a Tukey connection, and thus we will prove in this section that $\mathfrak{rr}^\perp \leq \mathrm{non}(\mathsf{null})$. The proof makes use of Rademacher's zero-one law (Theorem 1.2.18).

Theorem 3.3.8

$\mathrm{cov}(\mathsf{null}) \leq \mathfrak{rr}$ and $\mathrm{non}(\mathsf{null}) \geq \mathfrak{rr}^\perp$ ◁

Proof. We will give a Tukey connection $\varphi : \mathscr{R}_{io} \to \mathscr{C}_{\mathsf{null}}$ to show that $\mathrm{cov}(\mathsf{null}) \leq \mathfrak{rr}_{io}$ and $\mathfrak{rr}_{io}^\perp \leq \mathrm{non}(\mathsf{null})$. In this occasion we represent the reals as ${}^\omega 2$.

Naively, we could take $\varphi^- : {}^\omega 2 \to \textsc{ccs}$ to be the map $s \mapsto \left\langle (-1)^{s(n)}/n \;\middle|\; n \in \omega \right\rangle$. However, this sequence need not be conditionally convergent (although it will never be absolutely convergent). For example, the series for s being the zero map will diverge. Therefore, we take the map $s \mapsto \left\langle (-1)^{s(n)}/n \;\middle|\; n \in \omega \right\rangle$ when this is a conditionally convergent series, and $s \mapsto \langle (-1)^n/n \mid n \in \omega \rangle$ otherwise.

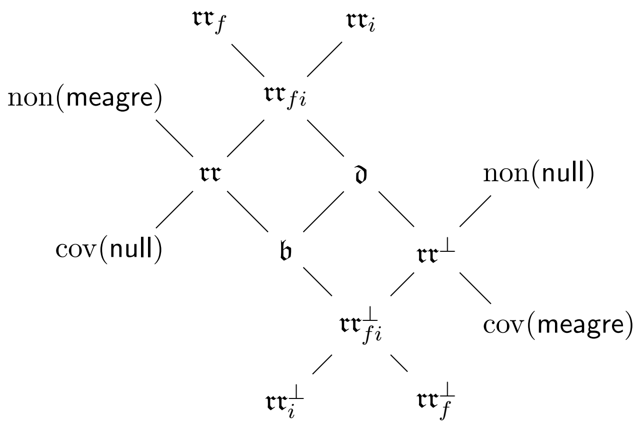
Similarly, we could naively define $\varphi^+ : \mathcal{S}(\omega) \to \mathsf{null}$ as the map that sends π to the set $B_\pi = \left\{ s \in {}^\omega 2 \;\middle|\; \sum_{n\in\omega} (-1)^{s(\pi(n))}/\pi(n) \text{ diverges} \right\}$, but we need to adjust for the special case we made for φ^-, so let $\varphi^+(\pi) = B_\pi \cup \left\{ s \in {}^\omega 2 \;\middle|\; \sum_{n\in\omega} (-1)^{s(n)}/n \text{ diverges} \right\}$ instead. It is a consequence of Rademacher's zero-one law that $\varphi^+(\pi) \in \mathsf{null}$.

We need to show that if we take an $s \in {}^{\omega}2$ and $\pi \in \mathcal{S}(\omega)$ and let $\overline{a} = \varphi^{-}(s)$ and $X = \varphi^{+}(\pi)$, then if $\sum_{n\in\omega} a_{\pi(n)}$ diverges to infinity or oscillates, then s must be in X. Suppose $\sum_{n\in\omega} a_{\pi(n)}$ does not converge, then there are two cases to consider.

If $\overline{a} = \left\langle (-1)^{s(n)}/n \;\middle|\; n \in \omega \right\rangle$, then if $\sum_{n\in\omega} a_{\pi(n)}$ diverges, by how B_π is defined, we get $s \in \varphi^{+}(\pi)$. Otherwise $\overline{a} = \langle (-1)^n/n \mid n \in \omega\rangle \neq \left\langle (-1)^{s(n)}/n \;\middle|\; n \in \omega \right\rangle$ and the latter diverges, meaning that $s \in \varphi^{+}(\pi)$. $\square$

3.4 Consistency of Strict Inequalities

To summarize the results so far, we can place the rearrangement numbers, the bounding and dominating numbers and the covering and uniformity numbers over the null and meagre ideals together in a Hasse diagram as follows:



We can say quite a lot about the consistency of strict inequalities between the rearrangement numbers and the other cardinal characteristics, based on the results from the last two sections.

Since $\mathfrak{rr}$ has two lower bounds, given by $\mathfrak{b}$ and $\mathrm{cov}(\mathsf{null})$, and it is both consistent that $\mathfrak{b} < \mathrm{cov}(\mathsf{null})$ (in the random model) and that $\mathrm{cov}(\mathsf{null}) < \mathfrak{b}$ (in the Hechler model), we see that $\mathfrak{rr}$ can be consistently strictly larger than either lower bound. The same can be said about the upper bounds of $\mathfrak{rr}^\perp$ being consistently strict, since it is both consistent that $\mathfrak{d} < \mathrm{non}(\mathsf{null})$ (in the dual random model) and that $\mathrm{non}(\mathsf{null}) < \mathfrak{d}$ (in the dual Hechler model).

We can also see that no relation between $\mathfrak{rr}$ and either of $\mathrm{cov}(\mathsf{meagre}) \leq \mathrm{non}(\mathsf{null})$ is provable, since it is consistent that $\mathrm{non}(\mathsf{null}) < \mathfrak{b} \leq \mathfrak{rr}$ (in the dual random model) and that $\mathfrak{rr} \leq \mathrm{non}(\mathsf{meagre}) < \mathrm{cov}(\mathsf{meagre})$ (in the Cohen model). Similarly, no relationship between $\mathfrak{rr}^\perp$ and either of $\mathrm{cov}(\mathsf{null}) \leq \mathrm{non}(\mathsf{meagre})$ is provable, since it is consistent that $\mathfrak{d} < \mathrm{cov}(\mathsf{null})$ (in the random model) and that $\mathrm{non}(\mathsf{meagre}) < \mathrm{cov}(\mathsf{meagre}) \leq \mathfrak{rr}^\perp$ (in the Cohen model).

As it is consistent that $\mathrm{non}(\mathsf{meagre}) = \mathfrak{b} < \mathfrak{d} = \mathrm{cov}(\mathsf{meagre})$ (in the Cohen model), we see that $\mathfrak{rr} < \mathfrak{rr}_{fi}$ and $\mathfrak{rr}^\perp_{fi} < \mathfrak{rr}^\perp$ are both consistent. This also shows the consistency of $\mathfrak{rr} < \mathfrak{rr}^\perp$. The reverse $\mathfrak{rr}^\perp < \mathfrak{rr}$ is consistent as well, since we already saw that $\mathfrak{d} < \mathrm{cov}(\mathsf{null})$ is consistent.

This means we are left with a few cases that we have not yet shown. First of all, there is the consistency of $\mathfrak{rr} < \mathrm{non}(\mathsf{meagre})$, which is unknown, as is the consistency of $\mathrm{cov}(\mathsf{meagre}) < \mathfrak{rr}^{\perp}$. Next, we have the consistency of $\mathfrak{rr}_{fi} < \mathfrak{rr}_f$ or $\mathfrak{rr}_{fi} < \mathfrak{rr}_i$ and dually of $\mathfrak{rr}_f^{\perp} < \mathfrak{rr}_{fi}^{\perp}$ and $\mathfrak{rr}_i^{\perp} < \mathfrak{rr}_{fi}^{\perp}$, which is also unknown.

Two characteristics that we have not yet seen in combination with the rearrangement numbers, are $\mathfrak{s}$ and $\mathfrak{r}$. It is consistent that $\mathfrak{rr} > \mathfrak{s}$ and $\mathfrak{rr}^{\perp} < \mathfrak{r}$, since it is consistent that $\mathrm{cov}(\mathsf{null}) > \mathfrak{s}$ and $\mathrm{non}(\mathsf{null}) < \mathfrak{r}$ (in the random model). Whether $\mathfrak{rr} < \mathfrak{s}$ is consistent is unknown, since it would also imply $\mathfrak{rr} < \mathrm{non}(\mathsf{meagre})$. We have $\mathfrak{rr} < \mathfrak{r}$ and $\mathfrak{rr}^{\perp} > \mathfrak{s}$ being consistent, since it is consistent that $\mathrm{non}(\mathsf{meagre}) < \mathfrak{r}$ and $\mathrm{cov}(\mathsf{meagre}) > \mathfrak{s}$ (in the Cohen model). Whether $\mathfrak{rr} > \mathfrak{r}$ is consistent is an open problem.

Finally there is the consistency of a strict upper bound $\mathfrak{rr}_i < \mathfrak{c}$ and $\mathfrak{rr}_f < \mathfrak{c}$. Both of these have been proved to be consistent in [4]. Using a dual forcing argument, we can use these proofs to show that $\aleph_1 < \mathfrak{rr}_i^{\perp}$ and $\aleph_1 < \mathfrak{rr}_f^{\perp}$ are consistent as well. This will be the subject of the following two sections.

Strict Bounds for Converging Rearrangement Numbers

In this section we will repeat the forcing argument from the rearrangement paper [4] that proves that $\mathfrak{rr}_f < \mathfrak{c}$ is consistent. We can use an iteration of the same forcing poset that is used in the paper to give a dual argument that $\mathfrak{rr}_f^{\perp} > \aleph_1$ is consistent as well. We will state the relevant lemmas with only a sketch of the proofs, and refer to section 8 of the rearrangement paper [4] for a detailed exposition.

The argument goes as follows. We start with a forcing poset, $\mathbb{P}_I$, such that if G is a generic filter for $\mathbb{P}_I$, then $\mathcal{M}[G]$ contains a generic permutation π for which every conditionally convergent series from the ground model will converge to a new limit.

Every ω-sequence of reals is essentially a real itself, since we have ${}^{\omega}({}^{\omega}\omega) \cong {}^{\omega\times\omega}\omega \cong {}^{\omega}\omega$. Therefore, using Theorem 1.3.34 we know that finite support iterations of $\mathbb{P}_I$ will not add any new conditionally convergent series at an uncountable limit step. We can employ this to see that the set consisting of the generic permutations added in each step of the iteration form a witness for $\mathfrak{rr}_f$. On the other hand, dually, we see that any set of conditionally convergent series that is already contained in an initial part of the iteration, will be made to converge to a new limit. This means that it can not be a witness for $\mathfrak{rr}_f^{\perp}$ in the model that is the result of the complete iteration.

Definition 3.4.1

Let $I \subseteq \textsc{ccs}$ be a set of conditionally convergent sequences and define the poset $\mathbb{P}_I$ as the subset of ${}^{<\omega}\omega \times [I]^{<\omega} \times \mathbb{Q}$ such that $\langle f, A, \varepsilon\rangle \in \mathbb{P}_I$ if and only if f is injective, $A = \left\{\overline{a}^1, \ldots, \overline{a}^n\right\}$ is nonempty, $\varepsilon > 0$ and for all $m \in \omega \setminus \mathrm{ran}(f)$ we have $\left|\left|\left\langle a_m^1, \ldots, a_m^n\right\rangle\right|\right| < \varepsilon / C_n$, that is, a vector consisting of the m-th element of the sequences in A lies within a ball at the origin of radius

ε/C_n. Here C_n is the constant from the Polygonal Confinement Theorem (Theorem 3.1.5) and $n = |A|$. We will without loss of generality pick an ordering of A and regard A as an infinite sequence of vectors $\overline{\boldsymbol{a}} = \langle\langle a_k^1, \ldots, a_k^n\rangle \mid k \in \omega\rangle : \omega \to \mathbb{R}^n$. As before, we will abbreviate $\mathrm{dom}(f)$ as n_f.

We define the ordering on $\mathbb{P}_I$ as $\langle g, B, \delta\rangle \leq \langle f, A, \varepsilon\rangle$ if and only if $f \subseteq g$, $A \subseteq B$, $\delta \leq \varepsilon$ and we have (with A represented as $\overline{\boldsymbol{a}}$) that

1. $\left\|\sum_{k\in[n_f,m)}\boldsymbol{a}_{g(k)}\right\| \leq \varepsilon$ for every $m \leq n_g$
2. $\left\|\sum_{k\in[n_f,n_g)}\boldsymbol{a}_{g(k)}\right\| \leq \varepsilon - \delta$ ◁

Lemma 3.4.2
$\langle\mathbb{P}_I, \leq\rangle$ is a forcing poset. ◁

There is no greatest element in $\mathbb{P}_I$. However, this is only a trivial problem: we can imagine adding an artifical top element $\mathbb{1}$ above all conditions, and work with $\mathbb{P}_I \cup \{\mathbb{1}\}$ instead.

Intuitively we can see the conditions $\langle f, A, \varepsilon\rangle \in \mathbb{P}_I$ as follows: f is an approximation for the generic permutation π in the sense that $\langle f, A, \varepsilon\rangle \Vdash \dot{\pi} \restriction n_f = f$. Hence f can be viewed as an initial segment of the permutation π.

The set A gives us an infinite sequence $\overline{\boldsymbol{a}}$ of n-dimensional vectors, such that the m-th terms lying outside of the range of f are very small. The choice of the bound ε/C_n is chosen such that any finite set of terms from the sequence $\overline{\boldsymbol{a}}$ can be rearranged suitably using the Polygonal Confinement Theorem, in such they form a polygonal path that stays within a ball of radius $(\varepsilon/C_n) \cdot C_n = \varepsilon$. This is to make sure that when we extend $\langle f, A, \varepsilon\rangle$ to a stronger condition $\langle g, A, \delta\rangle$, then the partial sums $\sum_{k<m}\boldsymbol{a}_{g(k)}$ for any $m \in [n_f, n_g)$ stay within an ε-neighbourhood of the sum $\sum_{k<n_f}\boldsymbol{a}_{f(k)}$. This is what inequality 1. in the definition of the ordering ascertains.

Inequality 2. in the definition of the ordering makes sure that the ordering is transitive. If we extend $\langle f, A, \varepsilon\rangle \geq \langle g, B, \delta\rangle \geq \langle h, C, \eta\rangle$, then the partial sums $\sum_{k<m}\boldsymbol{a}_{h(k)}$ for any $m \in [n_g, n_h)$ stay with a δ-neighbourhood of the sum $\sum_{k<n_g}\boldsymbol{a}_{g(k)}$. By the last inequality $\sum_{k<n_g}\boldsymbol{a}_{g(k)}$ lies in an $(\varepsilon - \delta)$-neighbourhood of the sum $\sum_{k<n_f}\boldsymbol{a}_{f(k)}$, and thus the partial sums $\sum_{k<m}\boldsymbol{a}_{h(k)}$ for any $m \in [n_g, n_h)$ are still in an ε-neighbourhood of $\sum_{k<n_f}\boldsymbol{a}_{f(k)}$.

Since we want a generic filter to give us a permutation π, we want to be able to extend any condition $\langle f, A, \varepsilon\rangle$ to some condition $\langle g, B, \delta\rangle$ such that g is a better approximation to π. This means that we want to be able to increase the range of f to include any initial segment $[0, k)$ of ω, to ascertain that the generic function π is surjective. Furthermore, we want to be able to make δ significantly smaller than ε, since g binds the partial sums of the series in A to lie in a small δ-neighbourhood. Letting δ tend to zero will mean that the generic permutation π will make the series in A converge. Finally, we want to be able to enlarge our control over all the conditionally convergent series, and thus we want to be able to add new series to A. The following lemma states that this is possible.

Lemma 3.4.3
Let $I \subseteq$ CCS be a maximally independent set. If $\langle f, A, \varepsilon\rangle \in \mathbb{P}_I$, then for any $k \in \omega$ and $\bar{b} \in I$ there is some $\langle g, B, \delta\rangle \in \mathbb{P}_I$ extending $\langle f, A, \varepsilon\rangle$ such that $k \subseteq \operatorname{ran}(g)$, $\delta < \frac{1}{k}$ and $\bar{b} \in B$. $\triangleleft$

> *Sketch of a proof.* In the proof, one regards A and $A \cup \{\bar{b}\}$ as infinite series of vectors. Using the Lévy-Steinitz theorem, we could find a permutation compatible with f such that the series of the permuted terms outside of the range of f will converge to the zero vector. Because of the promise made in Definition 3.4.1 that $||\langle a_m^1, \ldots, a_m^n\rangle|| < \varepsilon/C_n$ for these terms with m outside $\operatorname{ran}(f)$, we could select an initial part of those terms that will contain all terms larger than δ/C_{n+1} and such that the total initial part lies within $\varepsilon - \delta$ of the sum of the terms in the range of f. Then, using the Polygonal Confinement theorem, these terms can be rearranged to stay within a ball of radius $(\varepsilon/C_n) \cdot C_n = \varepsilon$, such that also inequality 1. is observed.

This lemma implies that all ground model conditionally convergent series will converge under the generic permutation. To see this, first note that any conditionally convergent series $\bar{a}$ is either an element of I, or there is some finite set $A \subseteq I$ such that $A \cup \{\bar{a}\}$ is not independent (by maximality of I). In the latter case there is some linear combination of the series in $A \cup \{\bar{a}\}$ that is absolutely convergent. It follows that $\bar{a}$ is the linear combination of the sequences in A and some absolutely convergent sequence. Hence, if we know that all sequences in I converge under the generic permutation, then it follows that any conditionally convergent sequence from the ground model converges under the generic permutation.

Given any $\bar{a} \in I$ and condition $\langle f, A, \varepsilon\rangle$, we can find an extension $\langle g, B, \delta\rangle$ such that $\bar{a} \in B$, and then for any $\eta > 0$ we can find an extension $\langle h, B, \eta\rangle$ that promises that the generic permutation will keep the partial sums of the permuted sequence $\bar{a}_\pi$ bound within a certain η-neighbourhood. Therefore, given any $k \in \omega$ the conditions $\langle g, B, \delta\rangle$ for which $\bar{a} \in B$ and $\delta < \frac{1}{k}$ are a dense subset in $\mathbb{P}_I$. By genericity the generic filter will contain such conditions for any $\bar{a} \in I$ and $k \in \omega$, and thus every $\bar{a} \in I$ converges under the generic permutation.

Of course it is not enough to just guarantee convergence under the generic permutation; if it were, the identity permutation would have sufficed, since clearly all conditionally convergent series stay convergent under the identity permutation. We also need to show that the limit of each conditionally convergent series is different than its limit under the generic permutation. We can show an even stronger statement, namely that the permuted series will not converge to any real number from the ground model.

Lemma 3.4.4
Let $r \in \mathbb{R}$, let $\langle f, A, \varepsilon\rangle \in \mathbb{P}_I$ and let $\boldsymbol{s} \in \mathbb{R}^n$ with $n = |A|$, then there exists a condition $\langle g, A, \delta\rangle \leq \langle f, A, \varepsilon\rangle$ such that $\left|\left|r - \sum_{k\in[0,n_g)} \boldsymbol{s} \cdot \boldsymbol{a}_{g(k)}\right|\right| > \delta\, ||\boldsymbol{s}||$, where $\bar{\boldsymbol{a}}$ is the sequence of vectors corresponding to the set A. $\triangleleft$

This lemma implies that conditionally convergent sequence from the ground model do not converge to a real number in the ground model under the generic permutation. To see this, let $\bar{a}$ be a conditionally convergent series, then there is some $\boldsymbol{s} \in \mathbb{R}^n$, $A \in [I]^n$ and absolutely convergent series $\overline{c}$ such that $a_k + c_k = \boldsymbol{s} \cdot \boldsymbol{a}_k$ for each $k \in \omega$. Let $r' = \sum_{k\in\omega} a_k + c_k$. Since rearranging the terms of c_k does not change the value of $\sum_{k\in\omega} c_k$, because $\overline{c}$ is absolutely convergent, we could work with $r = \sum_{k\in\omega} a_k = r' - \sum_{k\in\omega} c_k$. Note that both r and r' are reals from the ground model. Let $t \in \mathbb{R} \cap \mathcal{M}$ be any real from the ground model. By Lemma 3.4.3 the set A has some condition $\langle f, A, \varepsilon\rangle$ in the generic filter G, and thus Lemma 3.4.4 tells us that G also contains some $\langle g, A, \delta\rangle \leq \langle f, A, \varepsilon\rangle$ such that $\left|\left|t - \sum_{k\in[0,n_g)} \boldsymbol{s} \cdot \boldsymbol{a}_{g(k)}\right|\right| > \delta \,||\boldsymbol{s}||$. Since the generic permutation extends g, and since $\langle g, A, \delta\rangle \Vdash \left|\left|\sum_{k\in\omega\setminus n_g} \boldsymbol{a}_{\dot\pi(k)}\right|\right| < \delta$ by the definition of the ordering of $\mathbb{P}_I$, we see that $\langle g, A, \delta\rangle \Vdash \left|\left|\sum_{k\in\omega} \boldsymbol{s} \cdot \boldsymbol{a}_{\dot\pi(k)}\right|\right| \neq t$.

As a conclusion of the last two density lemmas, we get the following lemma about the convergence of ground model conditionally convergent series in the generic extension after forcing with $\mathbb{P}_I$.

Lemma 3.4.5
Let $\mathcal{M}$ be a ctm, let $I \subseteq$ CCS be a maximally independent set, let $\mathbb{P}_I$ be the poset from Definition 3.4.1 and let G be a generic filter. Then for any $\overline{a} \in \mathrm{CCS} \cap \mathcal{M}$ we see that the generic permutation $\pi = \bigcup\{f \in {}^{<\omega}\omega \mid \langle f, A, \varepsilon\rangle \in G\}$ has a limit $\sum_{n\in\omega} a_{\pi(n)}$ that is not present in the ground model. ◁

In order to iterate forcing with $\mathbb{P}_I$ without collapsing cardinals, we need to know that $\mathbb{P}_I$ behaves nicely enough. This turns out to be the case, as $\mathbb{P}_I$ is ccc. We can therefore use finite support iteration of $\mathbb{P}_I$ without collapsing cardinals, as the whole iteration will also be ccc. This is the final ingredient we needed to prove our main result.

Theorem 3.4.6
It is consistent that $\mathfrak{rr}_f < \mathfrak{c}$. ◁

Proof. Let $\mathcal{M}$ be a ctm such that $\mathcal{M} \vDash \mathfrak{c} > \aleph_1$ and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1\rangle$ be an ω_1-iteration with finite support with generic filter G, where $\dot{\mathbb{Q}}_\alpha$ names a poset $\mathbb{P}_{I_\alpha}$ from Definition 3.4.1 with I_α some maximally independent set in $\mathcal{M}[G_\alpha]$.

Let π_α be the generic permutation for $\mathbb{P}_{I_\alpha}$ defined by $G(\alpha)$, then by Lemma 3.4.5 we see that all conditionally convergent series in $\mathcal{M}[G_\alpha]$ will converge to a new limit under permutation of π_α.

By Theorem 1.3.34 we know that the ω_1-th step of the iteration does not add any reals, and thus any real $r \in \mathcal{M}[G]$ has already been decided in $\mathcal{M}[G_\alpha]$ for some $\alpha < \omega_1$. Since any $\overline{a} \in$ CCS is coded by a real (for example by taking some bijection $f : \mathbb{R} \to (0, 1)$ and letting $\overline{a}$ be coded by the real $0.x_1^1 x_2^1 x_1^2 x_1^3 x_2^2 x_3^1 x_1^4 \cdots$ with x_j^i the j-th digit of $f(a_i)$) we see that if $\overline{a}$ is a conditionally convergent sequence in $\mathcal{M}[G]$, then $\overline{a} \in \mathcal{M}[G_\alpha]$ for some $\alpha < \omega_1$, and hence π_α will make $\overline{a}$ converge to a new limit.

From this it follows that $\{\pi_\alpha \mid \alpha < \omega_1\}$ witnesses the property of $\mathfrak{rr}_f$. Since cardinals are preserved, we see that $\mathcal{M}[G] \vDash \aleph_1 = \mathfrak{rr}_f < \mathfrak{c}$. □

Theorem 3.4.7

It is consistent that $\mathfrak{rr}_f^\perp > \aleph_1$. $\triangleleft$

Proof. Let $\mathcal{M}$ be a ctm and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ be an ω_2-iteration with finite support, G be $\mathbb{P}_{\omega_2}$-generic, and $\dot{\mathbb{Q}}_\alpha$ name some $\mathbb{P}_{I_\alpha}$ with I_α a maximally independent set in $\mathcal{M}[G_\alpha]$. By Theorem 1.3.34 we know that the ω_2-th step does not add any reals, thus any real $r \in \mathcal{M}[G]$ has already been decided in $\mathcal{M}[G_\alpha]$ for some $\alpha < \omega_2$.

Let $C \subseteq$ CCS be a set of conditionally convergent series with $|C| = \aleph_1$. Note that each $\overline{a} \in$ CCS can be coded by a real, hence we can work with the set C' of codes for the series in C. Enumerate C' as $C' = \{r_\alpha \mid \alpha < \omega_1\}$, and for each $\alpha < \omega_1$ find a $\beta_\alpha < \omega_2$ such that $r_\alpha \in \mathcal{M}[G_{\beta_\alpha}]$. Let $\beta = \sup\{\beta_\alpha \mid \alpha < \omega_1\}$, then $C' \subseteq \mathcal{M}[G_\beta]$, and by using the coding, also $C \subseteq \mathcal{M}[G_\beta]$. Since ω_2 is regular, we see that $\beta < \omega_2$.

Let π be the generic permutation after forcing with $\dot{\mathbb{Q}}_\beta$, resulting in the extension $\mathcal{M}[G_{\beta+1}]$. It follows from Lemma 3.4.5 that for each $\overline{b} \in C$ we have $\sum_{n\in\omega} b_{\pi(n)}$ converge to a real that did not exist in $\mathcal{M}[G_\beta]$. Therefore π makes all conditionally convergent series from C converge to a new limit.

This means that all $\aleph_1$-sized sets in $\mathcal{M}[G]$ of conditionally convergent series have a permutation π such that all series in the set converge to a limit that is different from their original limit, or in other words, $\mathcal{M}[G] \vDash \mathfrak{rr}_f^\perp > \aleph_1$. $\square$

Strict Bounds for Infinite Rearrangement Number

As with the previous section, we will repeat the forcing argument from the rearrangement paper [4] that proves that $\mathfrak{rr}_i < \mathfrak{c}$ is consistent. The proof for the consistency of $\mathfrak{rr}_i < \mathfrak{c}$ does a finite support iteration of length ω_1 of a forcing $\mathbb{P}$ that is σ-centred over a model where the continuum $\mathfrak{c} > \aleph_1$ is large. To prove that $\mathbb{P}$ has the necessary density properties, it is assumed that the ground model satisfies Martin's Axiom for σ-centred posets, which is the statement that every set $\mathcal{D}$ of dense subsets of a σ-centred poset $\mathbb{P}$ with $|\mathcal{D}| < \mathfrak{c}$ has a generic filter in the ground model, denoted $\mathsf{MA}(\sigma\text{-centred})$.

In iterating this forcing, we need to make sure that in each intermediate step $\mathsf{MA}(\sigma\text{-centred})$ holds. For this reason $\mathbb{P}$ itself is a two-step iteration, where the first step forces $\mathsf{MA}(\sigma\text{-centred})$ and the second step forces that there exists a permutation under which all ground model series diverge to infinity. As we know, there are no new reals added in the ω_1-th step of the iteration, thus each conditionally convergent series is present after forcing with some initial part of the iteration and consequently the generic permutation of the next step in the iteration will make this series diverge to infinity.

Since we are mainly interested in making the dual rearrangement number $\mathfrak{rr}_i^\perp$ large, we want to add a large number of generic permutations, such that for any set of conditionally convergent series of size $\aleph_1$ there is one of the generic permutations that makes all of them diverge to infinity. This time, we are in the fortunate position that we could ignore the size of $\mathfrak{c}$, and thus the MA(σ-centred) requirement becomes irrelevant: we could let the ground model satisfy CH and use the following lemma.

Lemma 3.4.8
If $\mathcal{M} \vDash \mathsf{CH}$, then $\mathcal{M} \vDash \mathsf{MA}(\sigma\text{-centred})$. $\triangleleft$

> *Proof.* Under CH the forcing axiom $\mathcal{M} \vDash \mathsf{MA}(\sigma\text{-centred})$ reduces to the statement that any countable set of dense subsets of a σ-centred poset has a generic filter, and the existence of generic filters for countable sets of dense subsets of is provable in ZFC for *any* ccc poset, including all σ-centred posets[1]. $\square$

It therefore follows that the intermediate steps in the iteration from the paper that are used to force MA(σ-centred) are unnecessary to do when we start the iteration with a model of CH.

We will not prove the following theorem, and instead refer to section 9 of the rearrangement paper [4].

Theorem 3.4.9
It is consistent that $\mathfrak{rr}_i < \mathrm{non}(\mathsf{null}) = \mathfrak{c}$. $\triangleleft$

To prove that $\mathfrak{rr}_i^\perp > \aleph_1$ is consistent, we can prove a stronger claim, that $\mathfrak{rr}_i^\perp > \mathrm{cov}(\mathsf{null})$ is consistent. For the rest of this section we will assume CH holds in the ground model and fix an enumeration $\langle \overline{a}^\beta \mid \beta < \omega_1 \rangle$ of all conditionally convergent sequences. We will use the following forcing poset.

Definition 3.4.10
For some subset $X \subseteq \omega_1$ we define $\mathbb{I}_X \subseteq {}^{<\omega}\omega \times [\omega_1]^{<\omega} \times \omega$ with $\langle f, A, k \rangle \in \mathbb{I}_X$ if and only if $f : n \to \omega$ is injective, and for all $\beta \in A$ we have

$$\textstyle\sum_{i<n} a^\beta_{f(i)} > k \text{ if } \beta \in X, \text{ and}$$
$$\textstyle\sum_{i<n} a^\beta_{f(i)} < -k \text{ if } \beta \notin X.$$

As before, we abbreviate $\mathrm{dom}(f)$ as n_f. The ordering on $\mathbb{I}_X$ is defined as $\langle g, B, m \rangle \le \langle f, A, k \rangle$ if $f \subseteq g$, $A \subseteq B$, $k \le m$ and we have for all $n \in [n_f, n_g)$ and $\beta \in A$ that

$$\textstyle\sum_{i<n} a^\beta_{g(i)} > k \text{ if } \beta \in X, \text{ and}$$
$$\textstyle\sum_{i<n} a^\beta_{g(i)} < -k \text{ if } \beta \notin X. \quad \triangleleft$$

Here we see the injective function $f : n \to \omega$ as an approximation of a generic permutation π extending f such that all conditionally convergent series from the ground model diverge under permutation with π. The purpose of X is that it decides which sequences will diverge to ∞ and

[1] In other words CH implies the even stronger Martin's Axiom MA.

which diverge to $-\infty$. The main point of the forcing is how this X is decided, since we need to choose it in such a way that we can prove the following density lemma.

Lemma 3.4.11

There exists some $X \subseteq \omega_1$ such that for any $\langle f, A, k\rangle \in \mathbb{I}_X$, $n \in \omega$ and $\beta \in \omega_1$ there exists $\langle g, B, m\rangle \leq \langle f, A, k\rangle$ with $n \subseteq \operatorname{dom}(g) \cap \operatorname{ran}(g)$, $\beta \in B$ and $m \geq n$. $\triangleleft$

If X is chosen as in the above lemma, and G is a generic filter for $\mathbb{I}_X$, then each f with $\langle f, A, k\rangle \in G$ is injective by definition, and since all conditions in G are comparable they agree on their common domain. By the above lemma we can furthermore see that any condition can be extended to contain any $n \in \omega$ in both its range and its domain, thus $\pi = \bigcup \{f \mid \langle f, A, k\rangle \in G\}$ is a permutation.

We also see that if $\langle f, A, k\rangle \in G$, $\beta \in A$ and $\dot\pi$ is a name for the generic permutation, then $\langle f, A, k\rangle \Vdash \sum_{n\in\omega} a^\beta_{\dot\pi(n)} = \infty$ if $\beta \in X$ and $\langle f, A, k\rangle \Vdash \sum_{n\in\omega} a^\beta_{\dot\pi(n)} = -\infty$ if $\beta \notin X$ by a similar density argument. Finally we can see from the lemma that G contains $\langle f, A, k\rangle$ with $\beta \in A$ for any $\beta < \omega_1$, and thus G contains a condition for any $\beta < \omega_1$ that forces that the conditionally convergent sequence from the ground model $\overline{a}^\beta$ becomes divergent to infinity under the generic permutation. This shows that π makes all conditionally convergent sequences from the ground model diverge to infinity.

To find a suitable $X \subseteq \omega_1$, we will divide the set of conditionally convergent sequences into equivalence classes, such that there are permutations that will change the behaviour for all sequences in a single equivalence class without affecting the behaviour of sequences in the other classes. Each equivalence class will be represented by the element that comes first in the sequence $\left\langle \overline{a}^\beta \mid \beta \in \omega_1 \right\rangle$, and given $\overline{a}^\beta \in \textsc{ccs}$, we let $\zeta(\beta) = \alpha$ if α is the least ordinal such that $\overline{a}^\alpha$ is equivalent to $\overline{a}^\beta$. We let the set $A = \zeta[\omega_1]$ be the set of indices of representative elements and we say α and β are in the same equivalence class if $\zeta(\alpha) = \zeta(\beta)$.

The equivalence classes are constructed recursively simultaneously with the set A and the function ζ by building a matrix of sets $\left\langle X^\beta_\alpha \;\middle|\; \alpha \in A \wedge \alpha \leq \beta < \omega_1 \right\rangle$. The sets have a few properties, namely:

1. Any two sets $X^\beta_{\alpha_1}$ and $X^\beta_{\alpha_2}$ with distinct $\alpha_1, \alpha_2 \in A \cap \beta + 1$ are almost disjoint (i.e. have finite intersection).
2. If $\alpha \leq \beta_2 < \beta_1$ and $\alpha \in A$, then $X^{\beta_1}_\alpha \subseteq^* X^{\beta_2}_\alpha$ (i.e. $X^{\beta_1}_\alpha$ is an almost subset, or a subset except for finitely many elements, of $X^{\beta_2}_\alpha$).
3. If $\zeta(\alpha) \neq \zeta(\beta)$, then $\sum_{n\in X^\gamma_{\zeta(\alpha)}} |a^\beta_n|$ is convergent for any $\gamma > \alpha$.
4. If $\alpha = \zeta(\beta)$, then $\sum_{n\in X^\gamma_\alpha} |a^\beta_n|$ is divergent for any $\gamma > \alpha$.
5. If $\alpha = \zeta(\beta)$, then either all a^β_n with $n \in X^\beta_\alpha$ are positive or all are negative.

The set $X \subseteq \omega_1$ that we need is defined by letting $\beta \in X$ if all a^β_n with $n \in X^\beta_{\zeta(\beta)}$ are positive and $\beta \notin X$ if they are all negative. By the point 5. of the above properties we see that X is completely determined by the matrix of sets.

The construction of this matrix is done by double recursion over both indices of the sets X_α^β. The construction is described in detail in the paper, but it is not particularly insightful for our purposes to repeat the argument here. We therefore refer to the paper for a detailed proof of Lemma 3.4.11 and turn our attention to showing how this lemma implies the consistency of $\mathfrak{rr}_i^\perp > \text{cov}(\mathsf{null})$.

To see that $\text{cov}(\mathsf{null})$ will stay small when we do a finite support iteration of $\mathbb{I}_X$, we note that $\mathbb{I}_X$ is σ-centred, and thus does not add random reals by Theorem 1.3.30.

Lemma 3.4.12
$\mathbb{I}_X$ is σ-centred. ◁

Proof. For each $f \in {}^{<\omega}\omega$ and $k \in \omega$ we get a centred set $\{\langle f, A, k\rangle \mid A \in [\omega_1]^{<\omega}\}$, since for a finite number of sets $A_1, \dots, A_n \in [\omega_1]^{<\omega}$ the conditions $\langle f, A_i, k\rangle$ have the common extension $\langle f, \bigcup A_i, k\rangle$. □

We are now ready to prove the main theorem.

Theorem 3.4.13
It is consistent that $\mathfrak{rr}_i^\perp > \text{cov}(\mathsf{null}) = \aleph_1$. ◁

Proof. Let $\mathcal{M} \vDash \mathsf{CH}$ be a ctm, and let $\mathbb{P}_{\omega_2} = \left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \;\middle|\; \alpha < \omega_2 \right\rangle$ be a finite support iteration of length ω_2, where $\dot{\mathbb{Q}}_\alpha$ names the poset $\mathbb{I}_{X_\alpha}$ from Definition 3.4.10 and where X_α for each $\alpha < \omega_2$ is an appropriate set such that it satisfies Lemma 3.4.11. To be more precise, if G is $\mathbb{P}_{\omega_2}$ generic, then in $\mathcal{M}[G_\alpha]$ we construct a set X_α such that X_α satisfies Lemma 3.4.11, and we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the forcing poset $\mathbb{I}_{X_\alpha}$ in $\mathcal{M}[G_\alpha]$.

Since $\mathcal{M} \vDash \text{cov}(\mathsf{null}) = \aleph_1$ and by Theorem 2.4.2 a finite support iteration of σ-centred forcings does not add random reals, it follows that $\mathcal{M}[G] \vDash \text{cov}(\mathsf{null}) = \aleph_1$.

To see that $\mathfrak{rr}_i^\perp = \aleph_2$, let $C \subseteq \textsc{ccs} \cap \mathcal{M}[G]$ be a set of conditionally convergent sequences with $|C| = \aleph_1$. Since each conditionally convergent sequence is a real, and by Theorem 1.3.34 no reals are added in the ω_2-th step of the iteration, we see that for some $\alpha < \omega_2$ we have $C \in \mathcal{M}[G_\alpha]$.

A generic filter $G(\alpha)$ for $\mathbb{I}_{X_\alpha}$ defines a permutation $\pi = \bigcup\{f \mid \langle f, A, k\rangle \in G(\alpha)\}$ such that for every $\overline{a} \in \textsc{ccs} \cap \mathcal{M}[G_\alpha]$ we have $\mathcal{M}[G_{\alpha+1}] \vDash \sum_{n\in\omega} a_{\pi(n)} = \pm\infty$.

It follows that C does not satisfy the requirements for $\mathfrak{rr}_i^\perp$, since there is a permutation $\pi \in \mathcal{S}(\omega) \cap \mathcal{M}[G_{\alpha+1}]$ such that all conditionally convergent sequences in C diverge to infinity under permutation with π. Therefore we conclude that $\mathcal{M}[G] \vDash \mathfrak{rr}_i^\perp = \aleph_2$. □

Chapter 4

The Subseries Numbers

The subseries numbers are cardinal characteristics in the same spirit as the rearrangement numbers. These numbers have been studied in [7] after initially being defined by Joel David Hamkins in answer to another question on MathOverflow [25].

4.1 Subseries

With the rearrangement numbers we took a look at how conditionally convergent series behave under permutations. Conditional convergence is different from absolute convergence in the sense that the sum of an absolutely convergent series is invariant under permutation. We could give another characterisation of conditional convergence using the following proposition.

Proposition 4.1.1
A convergent series $\sum_{n\in\omega} a_n$ is conditionally convergent if and only if there is some subset $A \in [\omega]^\omega$ such that $\sum_{n\in A} a_n$ is divergent. $\triangleleft$

Such a series $\sum_{n\in A} a_n$ is what we call a **subseries** of $\sum_{n\in\omega} a_n$. It will be convenient to work with sequences $\overline{a} = \langle a_n \mid n \in \omega\rangle \in {}^\omega\mathbb{R}$. We say $\langle b_n \mid n \in \omega\rangle$ is a subsequence of $\overline{a}$ if there is some injective order preserving function $f : \omega \to \omega$ such that $b_n = a_{f(n)}$ for all $n \in \omega$. It follows that $f[\omega] \in [\omega]^\omega$ and $\sum_{n\in\omega} b_n = \sum_{n\in f[\omega]} a_n$.

We have the following analogue of the Riemann rearrangement theorem, which we will call the subseries theorem:

Theorem 4.1.2 — *Subseries theorem*
If $\overline{a}$ is a conditionally convergent sequence, then there exist ...

- ... a subset $A_x \in [\omega]^\omega$ for any $x \in \mathbb{R}$ such that $\sum_{n\in A_x} a_n = x$,
- ... subsets $B^+, B^- \in [\omega]^\omega$ such that $\sum_{n\in B^+} a_n = \infty$ and $\sum_{n\in B^-} a_n = -\infty$,
- ... a subset $C \in [\omega]^\omega$ such that $\sum_{n\in C} a_n$ diverges by oscillation. $\triangleleft$

We can slightly strengthen the theorem by replacing $[\omega]^\omega$ with $[\omega]^\omega_\omega$ everywhere.

4.2 The Subseries Numbers

The subseries numbers are defined in the same manner as the rearrangement numbers were defined. We start with the most general one.

Definition 4.2.1 — *Subseries number*
Let $\mathfrak{ß}$ be the least cardinality of a set $\mathcal{A} \subseteq [\omega]^\omega$ such that for every conditionally convergent sequence $\overline{a}$ there is a subset $A \in \mathcal{A}$ for which the series $\sum_{n\in A} a_n$ does not converge. $\triangleleft$

Note that we ask for the subseries to be divergent, instead of just have a different limit. This is because we could otherwise take $\mathcal{A} = \{\omega \setminus \{n\} \mid n \in \omega\}$, which contains a set A for any conditionally convergent series $\overline{a}$ that makes the series converge to a different limit: simply let $A = \omega \setminus \{n\}$ for any n such that $a_n \neq 0$. Consequently, when we give more refined versions of the definition above, there is no subseries number analogous to $\mathfrak{rr}_f$.

Definition 4.2.2 — *Subseries numbers*
Let $\mathfrak{ß}_o$ and $\mathfrak{ß}_i$ be the least cardinalities of sets $\mathcal{A}_o, \mathcal{A}_i \subseteq [\omega]^\omega$ respectively such that for every conditionally convergent sequence $\overline{a}$ there is a subset $A_o \in \mathcal{A}_o$ and $A_i \in \mathcal{A}_i$ for which the series $\sum_{n\in A_o} a_n$ diverges by oscillation and $\sum_{n\in A_i} a_n$ diverges to infinity. $\triangleleft$

We can describe the subseries numbers as the norms of triples, and thus we can define dual subseries numbers. We will denote these as $\mathfrak{ß}^\perp$, like we did with the rearrangement numbers. The subseries numbers as defined in the paper [7] have the following triples:

Definition 4.2.3 — *Triples for the subseries numbers*
Let CCS be the set of conditionally convergent sequences. We define the following two triples:

$$\mathscr{S}_o = \langle \mathrm{CCS}, [\omega]^\omega, S_o\rangle\,, \quad \mathscr{S}_i = \langle \mathrm{CCS}, [\omega]^\omega, S_i\rangle \quad \text{where}$$
$$S_o = \left\{\langle \overline{a}, A\rangle \;\middle|\; \textstyle\sum_{n\in A} a_n \text{ diverges by oscillation}\right\}$$
$$S_i = \left\{\langle \overline{a}, A\rangle \;\middle|\; \textstyle\sum_{n\in A} a_n \text{ diverges to } +\infty \text{ or } -\infty\right\}$$

Furthermore we define $\mathscr{S}_{io} = \mathscr{S}_i \cup \mathscr{S}_o$. Then $||\mathscr{S}_o|| = \mathfrak{ß}_o$, $||\mathscr{S}_i|| = \mathfrak{ß}_i$ and $||\mathscr{S}_{io}|| = \mathfrak{ß}$. $\triangleleft$

Similar to the rearrangement numbers we see that this implies $\mathfrak{ß} \leq \mathfrak{ß}_o$ and $\mathfrak{ß} \leq \mathfrak{ß}_i$.

We can change the definition of the triples slightly by substituting the set $[\omega]^\omega$ of possible responses with the set $[\omega]^\omega_\omega$ instead. That way we get the following triples:

$$\mathscr{S}'_o = \left\langle \mathrm{CCS}, [\omega]^\omega_\omega, S'_o \restriction (\mathrm{CCS} \times [\omega]^\omega_\omega)\right\rangle,$$
$$\mathscr{S}'_i = \left\langle \mathrm{CCS}, [\omega]^\omega_\omega, S'_i \restriction (\mathrm{CCS} \times [\omega]^\omega_\omega)\right\rangle,$$
$$\mathscr{S}'_{io} = \mathscr{S}'_i \cup \mathscr{S}'_o.$$

By the following lemma the above triples and those from Definition 4.2.3 are equivalent.

Lemma 4.2.4
$\mathscr{S}_{io}$ is equivalent to $\mathscr{S}'_{io}$, $\mathscr{S}_i$ to $\mathscr{S}'_i$, and $\mathscr{S}_o$ to $\mathscr{S}'_o$, $\triangleleft$

> *Proof.* We give Tukey connections $\mathscr{S}_{io} \xrightarrow{\varphi} \mathscr{S}'_{io} \xrightarrow{\psi} \mathscr{S}_{io}$. Let φ^-, ψ^- and ψ^+ be simply the identity. We let φ^+ be the identity on coinfinite subsets of ω, and send cofinite subsets to any arbitrary coinfinite set. That ψ is a Tukey connection is trivial. To see that φ is a Tukey connection, note that for any $\overline{a} \in \textsc{ccs}$ and $X \in [\omega]^\omega$ that if $\sum_{n\in X} a_n$ diverges, then X cannot be cofinite, since a finite change to the infinite series cannot make it diverge. Therefore $\varphi^+(X) = X$, and thus $\sum_{n\in\varphi^+(X)} a_n$ diverges as well.
>
> The same Tukey connections can be used to prove the claims for $\mathscr{S}'_o$ and $\mathscr{S}'_i$. $\square$

The interesting thing about these alternative definitions, is that it leads to a more natural way to look at convergent subseries. By only considering coinfinite subsets of ω we effectively get rid of the pathological counterexamples where only a finite number of terms in a series is omitted in the sum, and thus it becomes an interesting question to formulate a subseries number for convergent subseries.

Definition 4.2.5 — *Convergent subseries number*
Let $\mathfrak{ß}_c = ||\mathscr{S}_c||$, $\mathfrak{ß}_f = ||\mathscr{S}_f||$ and $\mathfrak{ß}_e = ||\mathscr{S}_e||$ be the norms of the following triples:

$$\mathscr{S}_c = \langle \textsc{ccs}, [\omega]^\omega_\omega, S_c\rangle\,, \quad \mathscr{S}_f = \langle \textsc{ccs}, [\omega]^\omega_\omega, S_f\rangle\,, \quad \mathscr{S}_e = \langle \textsc{ccs}, [\omega]^\omega_\omega, S_e\rangle\,, \quad \text{where}$$

$$S_c = \left\{\langle \overline{a}, A\rangle \mid \textstyle\sum_{n\in A} a_n \text{ converges}\right\}$$

$$S_f = \left\{\langle \overline{a}, A\rangle \mid \textstyle\sum_{n\in A} a_n \text{ converges and } \sum_{n\in A} a_n \neq \sum_{n\in\omega} a_n\right\}$$

$$S_e = \left\{\langle \overline{a}, A\rangle \mid \textstyle\sum_{n\in A} a_n = \sum_{n\in\omega} a_n\right\}$$ $\triangleleft$

Once again we could combine the triples by taking the union of their relations to get more refined notions of subseries numbers. Hovever, we have not much to say about these apart from the overly obvious, thus we will not consider them. It is worthy to note that not all combinations result in cardinal characteristics of the continuum. For example, one can easily see that $||\mathscr{S}'_o \cup \mathscr{S}'_i \cup \mathscr{S}_c|| = 1$.

The subseries numbers that have been discussed so far can be drawn in a rather unexciting diagram as follows.

$$\begin{tikzcd}[column sep=small]
\mathfrak{ß}_o \arrow[dr, dash] & & \mathfrak{ß}_i \arrow[dl, dash] \\
& \mathfrak{ß} &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
\mathfrak{ß}_e \arrow[dr, dash] & & \mathfrak{ß}_f \arrow[dl, dash] \\
& \mathfrak{ß}_c &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \mathfrak{ß}_c^\perp \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{ß}_f^\perp & & \mathfrak{ß}_e^\perp
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& \mathfrak{ß}^\perp \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{ß}_i^\perp & & \mathfrak{ß}_o^\perp
\end{tikzcd}$$

We will look at the convergent subseries numbers in Section 4.4, but first we will spend the next section to dualise the results from the subseries paper [7].

4.3 Subseries Numbers and Other Cardinal Characteristics

Like the rearrangement numbers, there are several upper and lower bounds for the subseries numbers provable in ZFC. In this section we will discuss these bounds, in particular the relation between the subseries numbers, the splitting and reaping numbers and the covering and uniformity numbers over the null and meagre ideals. Finally we will discuss how the subseries numbers and the rearrangement numbers are connected to each other using the bounding and dominating numbers.

As with Section 3.3, all results from this section are based on the proofs from the paper [7]. The relevance of this thesis lies once again in formulating these results in terms of Tukey connections to find out if dual statements hold for the dual subseries numbers.

Splitting and Reaping

The proof that $\mathfrak{s} \leq \mathfrak{ß}$ is based on the fact that an infinite series can only diverge if it has infinitely many nonzero terms and the fact that a change of only finitely many terms cannot influence the convergence / divergence of a series. We can stretch out the nonzero terms of a conditionally convergent sequence by inserting zeroes between them. This gives a way to translate these ideas of infinite sets of nonzero terms to subsets of ω.

Theorem 4.3.1
$\mathfrak{s} \leq \mathfrak{ß}$ and $\mathfrak{ß}^\perp \leq \mathfrak{r}$. ◁

Proof. Let $\mathscr{S}$ be the triple from Proposition 2.2.3 with as norm the splitting number $\mathfrak{s}$. We will give a Tukey connection $\varphi : \mathscr{S}_{io} \to \mathscr{S}$, implying that $\mathfrak{s} \leq \mathfrak{ß}$ and $\mathfrak{ß}^\perp \leq \mathfrak{r}$.

In order to define $\varphi^- : [\omega]^\omega \to \text{CCS}$, for any $B \in [\omega]^\omega$ let $f_B : \omega \to B$ be the unique order isomorphism. Let $\overline{a} \in \text{CCS}$ be an arbitrary conditionally convergent sequence. Define $\overline{b} = \varphi^-(B)$ to be the sequence with $b_{f_B(n)} = a_n$ for all $n \in \omega$ and $b_k = 0$ for all other $k \notin B$. We define φ^+ to be the identity on $[\omega]^\omega$.

To see this is a Tukey connection, let $A, B \in [\omega]^\omega$. Let $\overline{b} = \varphi^-(B)$. If $\langle \overline{b}, A \rangle \in S_{io}$, then $\sum_{n\in A} b_n$ diverges. For this to diverge there must be infinitely many nonzero terms, but by how we defined φ^+ we see that this can only be the case if $A \cap B$ is infinite, as $b_n = 0$ if $n \notin B$. Furthermore, as $\sum_{n\in B} b_n = \sum_{n\in\omega} a_n$ converges, we see that $B \setminus A$ must be infinite as well, since otherwise $\sum_{n\in A} b_n$ differs from $\sum_{n\in B} b_n$ in only finitely many nonzero terms, and a finite number of different terms cannot influence the convergence or divergence of a series. Therefore $|B\cap A| = |B\setminus A| = \aleph_0$, and thus $\langle B, A\rangle \in S$. $\square$

Cardinal Functions over Ideals on the Reals

In this section we will look at the relation between the subseries numbers and the ideals null and meagre. We start with the null idea to show that $\mathrm{cov}(\mathsf{null}) \leq \mathfrak{ß}$. The proof is a consequence of Rademacher's zero-one law (Theorem 1.2.18). The proof is indeed very similar to the proof of $\mathrm{cov}(\mathsf{null}) \leq \mathfrak{rr}$ (Theorem 3.3.8).

Theorem 4.3.2
$\mathrm{cov}(\mathsf{null}) \leq \mathfrak{ß}$ and $\mathfrak{ß}^\perp \leq \mathrm{non}(\mathsf{null})$. $\triangleleft$

Proof. We have the triple with norm $\mathrm{cov}(\mathsf{null})$, given by $\mathscr{C}_{\mathsf{null}} = \langle {}^\omega 2, \mathsf{null}, \in\rangle$ and we will formulate a Tukey connection $\varphi : \mathscr{S}_{io} \to \mathscr{C}_{\mathsf{null}}$.

We define $\varphi^- : {}^\omega 2 \to \mathrm{CCS}$ as $f \mapsto \overline{a}$, where $a_n = \frac{(-1)^{f(n)}}{n}$. Given $f \in {}^\omega 2$ let $\varphi^-(f)(n)$ denote the n-th term of $\varphi^-(f)$. We define $\varphi^+ : [\omega]^\omega \to \mathsf{null}$ as $A \mapsto D_A$, where $D_A = \left\{f \in {}^\omega 2 \mid \sum_{n\in A}\varphi^-(f)(n) \text{ diverges}\right\}$.

We have to show that $D_A \in \mathsf{null}$ for φ^+ to be well-defined. For any $f \in {}^\omega 2$ we have $(\varphi^-(f)(n))^2 = \frac{1}{n^2}$, therefore $\sum_{n\in A}(\varphi^-(f)(n))^2$ is a subseries of the absolutely convergent series $\sum_{n\in\omega}\frac{1}{n^2}$. This implies that $\sum_{n\in A}(\varphi^-(f)(n))^2$ is also absolutely convergent for any f. By Rademacher's zero-one law (Theorem 1.2.18) we then see that $\mu(D_A) = 0$.

Let $f \in {}^\omega 2$ and $A \in [\omega]^\omega$, and let $\overline{a} = \varphi^-(f)$ and $D_A = \varphi^+(A)$. Suppose that $\langle \overline{a}, A\rangle \in S_{io}$, then $\sum_{n\in A} a_n$ diverges. This implies that $f \in D_A \in \mathsf{null}$ by definition of φ^+. Thus φ is indeed a Tukey connection. $\square$

For the next inequality, remember that any product space $S = \prod_{k\in\omega} n_k$ with each $n_k \in \omega$ having the discrete topology is homeomorphic to ${}^\omega 2$ (Proposition 1.2.10).

Theorem 4.3.3
$\mathrm{cov}(\mathsf{meagre}) \leq \mathfrak{ß}_i$ and $\mathfrak{ß}_i^\perp \leq \mathrm{non}(\mathsf{meagre})$. $\triangleleft$

Proof. Let $\langle I_k \mid k \in \omega\setminus\{0\}\rangle$ be the interval partition of ω with $|I_k| = 2k$ and such that $\max(I_k) + 1 = \min(I_{k+1})$ for all k, that is, $I_1 = [0, 2)$, $I_2 = [2, 6)$, $I_3 = [6, 12)$, and so on. For each $k \in \omega$, let $D_k = \{A \subseteq I_k \mid |A| = k\}$, and define $K = \prod_{k\in\omega} D_k$ to be the space with the product topology induced by giving each D_k the discrete topology. The space K is homeomorphic with ${}^\omega 2$ as a corollary of Proposition 1.2.10, since each D_k is a finite discrete space. We can therefore safely work with the triple $\mathscr{C}_{\mathsf{meagre}} = \langle K, \mathsf{meagre}, \in\rangle$ for $\mathrm{cov}(\mathsf{meagre})$.

We define a Tukey connection $\varphi : \mathscr{S}_i \to \mathscr{C}_{\mathsf{meagre}}$ as follows.

Let $\varphi^- : K \to \textsc{ccs}$ send $x \mapsto \overline{a}$ where for $n \in I_k$ we define $a_n = \frac{1}{k^2}$ if $n \in I_k \cap x(k)$ and $a_n = \frac{-1}{k^2}$ if $n \in I_k \setminus x(k)$. This is a conditionally convergent function, as the sum of the positive terms in each interval I_k is $\frac{k}{k^2} = \frac{1}{k}$, and thus the sum of the positive terms in all the intervals is the harmonic series. The partial sum up to $\max(I_k)$ for any k is equal to 0, and any longer partial sum therefore stays within a bound of $\frac{1}{k}$ around 0, making the series converge to 0.

Let $\varphi^+ : [\omega]^\omega \to \mathsf{meagre} \subseteq \mathcal{P}(K)$ send $A \mapsto C_+ \cup C_-$ where

$$\begin{aligned} C_+ &= \left\{ x \in K \mid \textstyle\sum_{n \in A} \varphi^-(x) \text{ diverges to } +\infty \right\}, \\ C_- &= \left\{ x \in K \mid \textstyle\sum_{n \in A} \varphi^-(x) \text{ diverges to } -\infty \right\}. \end{aligned}$$

We have to show that $C_+ \cup C_-$ is meagre in K. Suppose $x, y \in K$ and $x =^* y$ (i.e. $\forall^\infty(x(k) = y(k))$), then let $\overline{a} = \varphi^-(x)$ and $\overline{b} = \varphi^-(y)$. Take $m \in \omega$ such that $x(k) = y(k)$ for all $k > m$, then it follows that $a_n = b_n$ for each $n \in \bigcup_{k>m} I_k$. Because $\bigcup_{k>m} I_k$ is a cofinite subset of ω, we see that $\overline{a}$ and $\overline{b}$ only differ from each other in a finite initial segment. It then follows that $\sum_{n\in\omega} a_n = +\infty$ if and only if $\sum_{n\in\omega} b_n = +\infty$, and thus $x \in C_+$ if and only if $y \in C_+$. This means that C_+ is a tail set, and therefore it is either meagre or comeagre by the Baire category zero-one law (Theorem 1.2.20).

Let $h : K \to K$ be the homeomorphism that sends $x(k) \mapsto I_k \setminus x(k)$. In other words, since $x \in K$ specifies for each $k \in \omega$ a subset $x(k) \subseteq I_k$ with $|x(k)| = k$, and $|I_k| = 2k$, we invert this selection and let $h(x)$ specify for each $k \in \omega$ the set $I_k \setminus x(k)$, which has $|I_k \setminus x(k)| = k$ as well. Clearly for any $n \in I_k$ we have $n \in x(k)$ if and only if $n \notin h(x)(k)$, and thus if $\varphi^-(x) = \overline{a}$, then $\varphi^-(h(x)) = -\overline{a}$. Therefore $x \in C_+$ if and only if $h(x) \in C_-$. Since $C_+ \cap C_- = \varnothing$, we see that C_+ cannot be comeagre; if it were, then so would C_- be since h is a homeomorphism, and the intersection of two comeagre sets is nonempty. Since both C_+ and C_- are meagre, we see that $C_+ \cup C_-$ is meagre.

To conclude this proof, we show that $\varphi : \mathscr{S}_i \to \mathscr{C}_{\mathsf{meagre}}$ is a Tukey connection. Let $x \in K$ and $A \in [\omega]^\omega$, and let $\overline{a} = \varphi^-(x)$ and $C_+ \cup C_- = \varphi^+(A)$, then $\langle \overline{a}, A \rangle \in S_i$ if $\sum_{n\in A} a_n$ diverges to infinity. But by definition of C_+ and C_- it then follows that $x \in C_+$ or $x \in C_-$. Thus φ is a Tukey connection. $\square$

Lastly we have the relation between $\mathfrak{s}_o$ and the meagre ideal.

Theorem 4.3.4
$\mathfrak{s}_o \leq \operatorname{non}(\mathsf{meagre})$ and $\operatorname{cov}(\mathsf{meagre}) \leq \mathfrak{s}_o^\perp$. ◁

Proof. We work with the triple $\mathscr{N}_{\mathsf{meagre}} = \langle \mathsf{meagre}, [\omega]^\omega, \not\ni \rangle$, with norm $\operatorname{non}(\mathsf{meagre})$. Remember that $[\omega]^\omega$ inherits the topology from being a G_δ subset of $\mathcal{P}(\omega)$, which in turn receives its topology from the Cantor space ${}^\omega 2$ under the characteristic map

$\chi : X \in \mathcal{P}(\omega) \mapsto x \in {}^{\omega}2$ with $n \in X$ if and only if $x(n) = 1$.

We give a Tukey connection $\varphi : \mathscr{N}_{\mathsf{meagre}} \to \mathscr{S}_o$.

For $\overline{a} \in \textsc{ccs}$ define $\varphi^-(\overline{a})$ to be the set of all $A \in [\omega]^\omega$ such that $\sum_{n\in A} a_n$ does not diverge by oscillation. We have to prove that $\varphi^-(\overline{a})$ is meagre, thus we will show that the set $O_{\overline{a}} = [\omega]^\omega \setminus \varphi^-(\overline{a})$ of all sets $A \in [\omega]^\omega$ for which $\sum_{n\in A} a_n$ diverges by oscillation is comeagre.

We define two subsets of $[\omega]^\omega$:

$$U_k = \left\{A \in [\omega]^\omega \mid \exists m \in \omega \left(\textstyle\sum_{n\in A\cap m} a_n \geq k\right)\right\}$$

$$V_k = \left\{A \in [\omega]^\omega \mid \exists m \in \omega \left(\textstyle\sum_{n\in A\cap m} a_n \leq -k\right)\right\}$$

Let $A \in U_k$, and take $m \in \omega$ such that $\sum_{n\in A\cap m} a_n \geq k$. It is easy to see that for any $B \in [\omega]^\omega$ with $A \cap m = B \cap m$ we also have $B \in U_k$. The set of all such B is a basic open in the topology of $[\omega]^\omega$, and thus every $A \in U_k$ is part of an open neighbourhood contained in U_k. This shows U_k (and similarly V_k) is open.

Furthermore, let $B \in [\omega]^\omega$ and consider any basic open $U \supseteq B$, then U is of the form $U = \{C \in [\omega]^\omega \mid C \cap m = S\}$ for some $S \subseteq m \in \omega$. As $\overline{a}$ is conditionally convergent, we could find some set $C \in [\omega]^\omega$ for which $\sum_{n\in C} a_n = +\infty$. Let $A = S \cup (C \cap (\omega \setminus m))$, then we still have $\sum_{n\in A} a_n = +\infty$, and thus $A \in U_k$. This shows U_k (and similarly V_k) is dense.

Let $\mathcal{O} = \bigcap_{k\in\omega} U_k \cap V_k$, then $\mathcal{O}$ is a countable intersection of dense open sets, and therefore $\mathcal{O}$ is comeagre. It is easy to see that if $A \in \mathcal{O}$, then $\sum_{n\in A} \overline{a}$ diverges by oscillation, thus $\mathcal{O} \subseteq O_{\overline{a}}$. It follows $O_{\overline{a}}$ is comeagre as well.

We define φ^+ as the identity. To see that φ is a Tukey connection, let $\overline{a} \in \textsc{ccs}$ and $A \in [\omega]^\omega$. If $A \notin \varphi^-(\overline{a})$, then $A \in O_{\overline{a}}$, and thus A diverges by oscillation. □

Bounding, Dominating and Rearrangement Numbers

As the last part of this section we will discuss the dualisation of a relation between the subseries numbers and the rearrangement numbers using the bounding and dominating numbers. The idea is based on the following.

Suppose $A \subseteq [\omega]^\omega_\omega$ is a set such that for any $\overline{a} \in \textsc{ccs}$ there is some $X \in A$ such that $\sum_{n\in X} a_n$ diverges. Then we could find some positive $c \in \mathbb{R}_{>0}$ and some sequence of intervals $[m_i, m_{i+1})$ with $m_i < m_{i+1}$ for all i such that $\left|\sum_{n\in[m_i,m_{i+1})\cap X} a_n\right| > c$. We could now use X to define a permutation π for which $\sum_{n\in\omega} a_{\pi(n)}$ diverges, namely by letting $\pi \restriction [m_i, m_{i+1})$ be a bijection on $[m_i, m_{i+1})$ such that if $n \in [m_i, m_{i+1}) \cap X$ and $n' \in [m_i, m_{i+1}) \setminus X$, then we have $\pi(n) < \pi(n')$. It is easy to see that $\sum_{n\in\omega} a_{\pi(n)}$ then diverges.

The problem is that this π depends not only on the set $X \in A$, but also on the conditionally convergent sequence $\overline{a}$. We can therefore not uniformly give a way to translate this X into a suitable permutation. The solution is to use bounding and dominating subsets of $[\omega]^\omega_\omega$, as we will see in the proof of the next theorem.

Theorem 4.3.5

$\mathfrak{rr} \leq \max\{\mathfrak{b}, \mathfrak{F}\}$, $\mathfrak{rr}_i \leq \max\{\mathfrak{d}, \mathfrak{F}_i\}$ and $\min\{\mathfrak{d}, \mathfrak{F}^\perp\} \leq \mathfrak{rr}^\perp$, $\min\{\mathfrak{b}, \mathfrak{F}_i^\perp\} \leq \mathfrak{rr}_i^\perp$. $\triangleleft$

Proof. We use the triple $\overline{\mathscr{B}} = \langle [\omega]^\omega_\omega, [\omega]^\omega_\omega, \overline{B} \rangle$ from Proposition 2.3.6 and its dual $\overline{\mathscr{D}} = \langle [\omega]^\omega_\omega, [\omega]^\omega_\omega, \overline{D} \rangle = \overline{\mathscr{B}}^\perp$, with norm $\left|\left|\overline{\mathscr{D}}\right|\right| = \mathfrak{d}$. We will first give a Tukey connection $\varphi : \mathscr{S}'_{io} {}^\frown \overline{\mathscr{B}} \to \mathscr{R}_{io}$ to show that $\mathfrak{rr} \leq \max\{\mathfrak{F}, \mathfrak{b}\}$ and $\min\{\mathfrak{F}^\perp, \mathfrak{d}\} \leq \mathfrak{rr}^\perp$. Afterwards we will give a closely related Tukey connection $\psi : \mathscr{S}'_i {}^\frown \overline{\mathscr{D}} \to \mathscr{R}_i$ to show that $\mathfrak{rr}_i \leq \max\{\mathfrak{d}, \mathfrak{F}_i\}$ and $\min\{\mathfrak{b}, \mathfrak{F}_i^\perp\} \leq \mathfrak{rr}_i^\perp$.

We have $\varphi^- : \textsc{ccs} \to \textsc{ccs} \times {}^{[\omega]^\omega_\omega}[\omega]^\omega_\omega$ sending $\overline{a}$ to $(\overline{a}, Z_{(\cdot)})$. To define Z_X for $X \in [\omega]^\omega_\omega$, we first define a strictly increasing sequence $\overline{m} \in \uparrow({}^\omega X)$. Consider two cases: if $\sum_{n\in X} a_n$ converges, let $\overline{m}$ be arbitrary, and if $\sum_{n\in X} a_n$ diverges, choose a suitable positive constant $c \in \mathbb{R}_{>0}$ to define $\overline{m}$ such that for all $k \in \omega$ we have $\left|\sum_{n\in[m_k,m_{k+1})\cap X} a_n\right| > c$. We define $Z_X = \{|m_k \cap X| \mid k \in \omega\}$. We will denote $m'_k = |m_k \cap X|$, or in other words, $\langle m'_k \mid k \in \omega\rangle$ is an increasing enumeration of Z_X.

For $X, Y \in [\omega]^\omega_\omega$ define $\pi_{X,Y} : \omega \to \omega$ to be the unique map such that $\pi_{X,Y} \restriction X$ is an order-preserving bijection onto Y and $\pi_{X,Y} \restriction (\omega \setminus X)$ is the order-preserving bijection onto $\omega \setminus Y$. Let $f_Y : \omega \to Y$ be an order isomorphism, then define the set $\widetilde{Y} = \{f_Y(n) + n \mid n \in \omega\}$. We define $\varphi^+ : [\omega]^\omega_\omega \times [\omega]^\omega_\omega \to \mathcal{S}(\omega)$ to be the map $(X, Y) \mapsto \pi_{\omega\setminus\widetilde{Y}, X}$.

Suppose that $\overline{a} \in \textsc{ccs}$ and $X, Y \in [\omega]^\omega_\omega$ and let $\varphi^-(\overline{a}) = (\overline{a}, Z_{(\cdot)})$ and $\varphi^+(X, Y) = \sigma$, where $\sigma = \pi_{\omega\setminus\widetilde{Y}, X}$. Let $\overline{x} = \langle x_i \mid i \in \omega\rangle$ and $\overline{y} = \langle y_i \mid i \in \omega\rangle$ be strictly increasing enumerations of X and Y. We will write $\widetilde{y}_i = f_Y(i) = y_i + i$ for the elements of $\widetilde{Y}$.

If $\langle(\overline{a}, Z_{(\cdot)}), (X, Y)\rangle$ satisfies the relation of $\mathscr{S}'_{io} {}^\frown \overline{\mathscr{B}}$, then we see that $\sum_{n\in X} a_n$ diverges and thus Z_X is defined from $\overline{m}$ in the non-arbitrary way. We furthermore see that Z_X contains infinitely many $x < y$ such that $[x, y) \cap Y = \varnothing$. We have to show that $\sum_{n\in\omega} a_{\sigma(n)}$ diverges.

There are infinitely many $k \in \omega$ such that $[m'_k, m'_{k+1}) \cap Y = \varnothing$, since $Z_X = \{m'_k \mid k \in \omega\}$. Fix one of such k and let $l \in \omega$ be such that $y_l < m'_k < m'_{k+1} \leq y_{l+1}$, then we see that $\widetilde{y}_l = y_l + l < m'_k + l < m'_{k+1} + l < y_{l+1} + l + 1 = \widetilde{y}_{l+1}$. Now note that if $\widetilde{y}_l < n < \widetilde{y}_{l+1}$, then $\sigma(n) = x_{n-l}$, since n is the $(n-l)$-th element of $\omega \setminus \widetilde{Y}$. This implies that $\sigma \restriction [m'_k + l, m'_{k+1} + l)$ is an order isomorphism with $[x_{m'_k}, x_{m'_{k+1}}) \cap X$. But $x_{m'_k} = m_k$, and thus $\sigma^{-1}[[m_k, m_{k+1}) \cap X] = [m'_k + l, m'_{k+1} + l)$ is an interval.

From this it follows that $\sum_{n\in[m'_k+l, m'_{k+1}+l)} a_{\sigma(n)} = \sum_{n\in[m_k,m_{k+1})\cap X} a_n$, and by the construction of $\overline{m}$ we know that this has an absolute value larger than c. Therefore

$\sum_{n\in\omega} a_{\sigma(n)}$ contains infinitely many intervals with a sum larger than some fixed positive value c, which implies that $\sum_{n\in\omega} a_{\sigma(n)}$ diverges.

As for ψ, it can be defined exactly as φ is. In this case we have the assumption that $\langle(\overline{a}, Z_{(\cdot)}), (X, Y)\rangle$ satisfies the relation of $\mathscr{S}_i'{}^\frown\overline{\mathscr{D}}$, and thus $\sum_{n\in X} a_n$ diverges to infinity and for almost all $x < y$ in Y we have $[x, y) \cap Z_X \neq \varnothing$. This means that, except for finitely many $k \in \omega$, between any two consecutive elements m_k', m_{k+1}' of Z_X we can have at most one point in Y lying in the interval $[m_k', m_{k+1}')$.

Since $\sum_{n\in\omega} a_n$ is a convergent series, for almost all n we have $|a_n| < \frac{c}{2}$. We can therefore pick some $K \in \omega$ such that for all $k > K$ we have $[m_k', m_{k+1}')$ contain either no elements of Y, or a single element $n_k \in Y$ such that $\left|a_{\sigma(n_k)}\right| < \frac{c}{2}$. In the first case we have by similar reasoning as before that $\left|\sum_{n\in[m_k'+l, m_{k+1}'+l)} a_{\sigma(n)}\right| > c$, and in the second case we have $\left|\sum_{n\in[m_k'+l, m_{k+1}'+l)} a_{\sigma(n)}\right| > \frac{c}{2}$, since the term $a_{\sigma(n_k)}$ could change the sum with at most $\frac{c}{2}$.

Finally to see that $\sum_{n\in\omega} a_{\sigma(n)}$ diverges towards infinity, note that we assumed that $\sum_{n\in\omega} a_n$ diverges to infinity. This means that when we chose the sequence $\overline{m}$ such that $\left|\sum_{n\in[m_k, m_{k+1})\cap X} a_n\right| > c$, we could actually choose $\overline{m}$ is such a way that we have $\sum_{n\in[m_k, m_{k+1})\cap X} a_n > c$ for almost all k, or $\sum_{n\in[m_k, m_{k+1})\cap X} a_n < -c$ for almost all k. This implies that also $\sum_{n\in\omega} a_{\sigma(n)}$ diverges to infinity. $\square$

Unlike the rearrangement numbers, we do not know whether $\mathfrak{ß}_o = \mathfrak{ß}$ is provable, nor do we know this about their duals. We can however prove a weaker claim, that $\mathfrak{ß}_o$ is bound above by $\mathfrak{ß}$ as long as $\mathfrak{b}$ is small. Similar to the rearrangement numbers, though, is that the proof given in the subseries paper [7] does not immediately translate to a Tukey connection, since it splits the set of conditionally convergent series into those for which a subseries diverges by oscillation and those for which a subseries diverges to infinity, and treats both cases separately. This cannot be dualised, for similar reasons as we saw in Example 3.3.3.

Fortunately we can give a variation of the proof in the paper that *does* work for our purposes. We make use of the following lemma.

Lemma 4.3.6

Let $\overline{a}$ be a conditionally convergent sequence and $A \in [\omega]^\omega$. If $\sum_{n\in A} a_n$ diverges, there is a positive real number $c \in \mathbb{R}_{>0}$ for which there are infinitely many disjoint intervals $I_i \subseteq \omega$ such that one of the following is true:

- $\sum_{n\in I_i\setminus A} a_n < -c$ and $c < \sum_{n\in I_i\cap A} a_n$ for all $i \in \omega$, or
- $\sum_{n\in I_i\cap A} a_n < -c$ and $c < \sum_{n\in I_i\setminus A} a_n$ for all $i \in \omega$. $\triangleleft$

Proof. We know that $\sum_{n\in\omega} a_n$ converges, thus for any $\varepsilon > 0$ we could find some $m' \in \omega$ for which $\sum_{m'\leq n\leq m} a_n \in (-\varepsilon, \varepsilon)$ for every $m > m'$.

If $\sum_{n\in A} a_n$ diverges by oscillation, then there are $d \in \mathbb{R}$ and $c \in \mathbb{R}_{>0}$ for which $\sum_{n\in A\cap m} a_n < d$ for infinitely many $m \in \omega$ and $\sum_{n\in A\cap m} a_n > d + 2c$ for infinitely many $m \in \omega$. Fix $\varepsilon < c$ and $m' \in \omega$ as above. For all $i \in \omega$ pick m_i, k_i with $m' < m_0$ and $m_i < k_i < m_{i+1}$, such that $\sum_{n \in A \cap m_i} a_n < d$ and $\sum_{n \in A \cap k_i} a_n > d + 2c$. It follows from this that $\sum_{n \in A \cap [m_i, k_i)} a_n > 2c$ and $\sum_{n \in [m_i, k_i)} a_n < \varepsilon < c$ (since $m_i > m'$), and thus $\sum_{n \in [m_i, k_i) \setminus A} a_n < -c$. We can therefore take $I_i = [m_i, k_i)$ for each $i \in \omega$ to see that the first bullet point from the lemma holds.

If $\sum_{n \in A} a_n$ diverges to $+\infty$, then we could take any $c \in \mathbb{R}_{>0}$ and fix some $\varepsilon < c$ and m' as above. We could pick for all $i \in \omega$ some m_i with $m' < m_0$ and $m_i < m_{i+1}$ such that $\sum_{n \in A \cap [m_i, m_{i+1})} a_n > 2c$. That $\sum_{n \in [m_i, m_{i+1}) \setminus A} a_n < c$ follows from the same reasons as with the case for oscillating subseries. Therefore we take $I_i = [m_i, m_{i+1})$ for each $i \in \omega$ to see that the first bullet point from the lemma holds.

Finally if $\sum_{n \in A} a_n$ diverges to $-\infty$, we have the same argument as when it diverges to $+\infty$, except we now let $\sum_{n \in A \cap [m_i, m_{i+1})} a_n < -2c$ and get $\sum_{n \in [m_i, m_{i+1}) \setminus A} a_n > c$, meaning that the second bullet point from the lemma holds instead. $\square$

Theorem 4.3.7
$\mathfrak{ß}_o \leq \max\{\mathfrak{b}, \mathfrak{ß}\}$ and $\min\{\mathfrak{d}, \mathfrak{ß}^\perp\} \leq \mathfrak{ß}_o^\perp$. $\triangleleft$

Proof. We will use that $\mathfrak{s} \leq \mathfrak{b}$ and give a Tukey connection $\varphi : (\mathscr{S}'_{io} {}^\frown \overline{\mathscr{B}}) {}^\frown \mathscr{S} \to \mathscr{S}'_o$. We have the following triple for $(\mathscr{S}'_{io} {}^\frown \overline{\mathscr{B}}) {}^\frown \mathscr{S}$:

$$\left\langle \text{CCS} \times {}^{[\omega]^\omega_\omega}[\omega]^\omega_\omega \times {}^{[\omega]^\omega_\omega \times [\omega]^\omega_\omega}[\omega]^\omega, \quad [\omega]^\omega_\omega \times [\omega]^\omega_\omega \times [\omega]^\omega, \quad Z \right\rangle$$

Here Z is the relation such that $(\overline{a}, f, g)\ Z\ (A, B, S)$, with $f : [\omega]^\omega_\omega \to [\omega]^\omega_\omega$ and $g : [\omega]^\omega_\omega \times [\omega]^\omega_\omega \to [\omega]^\omega$ if and only if $\sum_{n \in A} a_n$ diverges, there are infinitely many $x < y$ in $f(A)$ such that $[x, y) \cap B = \varnothing$ and $g(A, B)$ is split by S.

The function $\varphi^+ : [\omega]^\omega_\omega \times [\omega]^\omega_\omega \times [\omega]^\omega \to [\omega]^\omega_\omega$ is the easy part to describe: let $h_B : \omega \to B$ be the order isomorphism for the set B, then we define φ^+ as:

$$\varphi^+(A, B, S) = \left(\bigcup_{n \in S} (h(n), h(n+1)] \cap A \right) \cup \left(\bigcup_{n \notin S} (h(n), h(n+1)] \setminus A \right).$$

To define $\varphi^- : \text{CCS} \to \text{CCS} \times {}^{[\omega]^\omega_\omega}[\omega]^\omega_\omega \times {}^{[\omega]^\omega_\omega \times [\omega]^\omega_\omega}[\omega]^\omega$, we let $\varphi^-(\overline{a}) = (\overline{a}, f, g)$, where $f : [\omega]^\omega_\omega \to [\omega]^\omega_\omega$ and $g : [\omega]^\omega_\omega \times [\omega]^\omega_\omega \to [\omega]^\omega$ will be defined below.

Let $A, B \in [\omega]^\omega_\omega$. If $\sum_{n \in A} a_n$ converges, we take any arbitrary value for $f(A)$ and $g(A, B)$, as this case will be irrelevant. Therefore, assume that $\sum_{n \in A} a_n$ diverges. By Lemma 4.3.6 we can find a family of disjoint intervals $\{I_i \mid i \in \omega\}$ and a positive real number $c \in \mathbb{R}_{>0}$ such that $\sum_{n \in I_i \setminus A} a_n < -c$ and $c < \sum_{n \in I_i \cap A} a_n$ for all i or such that $\sum_{n \in I_i \cap A} a_n < -c$ and $c < \sum_{n \in I_i \setminus A} a_n$ for all i. We will assume without loss of generality that the first is the case and we will also assume that $\min(I_i) < \min(I_{i+1})$ for all i. We let $f(A) = \{\min(I_i) \mid i \in \omega\}$ in this case. Clearly $f(A)$ is infinite, and it follows that $f(A)$ is coinfinite from the fact that $\bar{a}$ is conditionally convergent, and thus not all I_i could be singleton sets.

In case there are only finitely many $x < y \in f(A)$ such that $[x, y) \cap B = \varnothing$, we define $g(A, B)$ to be arbitrary, as once again this case will be irrelevant. Therefore let us assume that $\sum_{n \in A} a_n$ diverges and that there are infinitely many $x < y \in f(A)$ such that $[x, y) \cap B = \varnothing$. Let $h : \omega \to B$ be the order isomorphism for the set B, then define $g(A, B) = \{n \in \omega \mid \exists x, y \in f(A)(x < y \wedge [x, y) \subseteq (h(n), h(n+1)])\}$. If $x < y \in f(A)$ and $[x, y) \cap B = \varnothing$, then $h(n) < x < y \leq h(n+1)$ for some $n \in \omega$, thus we see that $g(A, B)$ is indeed infinite.

Now to see that φ is indeed a Tukey connection, let $\bar{a} \in \text{CCS}$, $A, B \in [\omega]^\omega_\omega$ and $S \in [\omega]^\omega$. We let $\varphi^-(\bar{a}) = (\bar{a}, f, g)$ and $\varphi^+(A, B, S) = C$. Suppose that $(\bar{a}, f, g)$ Z (A, B, S), then $\sum_{n \in A} a_n$ diverges, thus f is defined from the intervals I_i, as above. We also see that there are infinitely many $x < y \in f(A)$ such that $[x, y) \cap B = \varnothing$, thus $g(A, B)$ is defined as above as well from the order isomorphism $h : \omega \to B$. Finally we have that $g(A, B)$ is split by S, thus $g(A, B) \cap S$ and $g(A, B) \setminus S$ are both infinite. We will show that there is a $c \in \mathbb{R}_{>0}$ such that $\sum_{n \in I \cap C} a_n > c$ for infinitely many nonempty intervals I, and $\sum_{n \in I \cap C} a_n < -c$ for infinitely many nonempty intervals I. This implies that $\sum_{n \in C} a_n$ diverges by oscillation.

For infinitely many $n \in S$ we have $(h(n), h(n+1)] \supseteq [x, y)$ for some $x < y \in f(A)$, since $g(A, B) \cap S$ is infinite. Without loss of generality we can let $x = \min(I_i)$ and $y = \min(I_{i+1})$, which shows that $[x, y) \supseteq I_i$. It then follows that $I_i \cap C = I_i \cap A$ from the definition of C. By how we defined I_i, we know that $\sum_{n \in I_i \cap A} a_n > c$, and thus $\sum_{n \in I_i \cap C} a_n > c$ for infinitely many i.

On the other hand, $g(A, B) \setminus S$ is infinite as well, thus there are infinitely many $n \notin S$ for which $(h(n), h(n+1)] \supseteq [x, y)$ for some $x < y \in f(A)$. Similar to the other case, we see that from this it follows that $I_i \cap C = I_i \setminus A$ for infinitely many i, and thus by how I_i is defined, we know that this implies that $\sum_{n \in I_i \cap C} a_n < -c$ for infinitely many i.

Since $\left|\left|(\mathscr{S}'_{io} {}^\frown \mathscr{B}) {}^\frown \mathscr{S}\right|\right| = \max\{\mathfrak{ß}, \mathfrak{b}, \mathfrak{s}\}$, which equals $\max\{\mathfrak{ß}, \mathfrak{b}\}$ by $\mathfrak{s} \leq \mathfrak{b}$, we see that this proves $\mathfrak{ß}_o \leq \max\{\mathfrak{ß}, \mathfrak{b}\}$, and dually that $\mathfrak{ß}_o^\perp \geq \min\{\mathfrak{ß}^\perp, \mathfrak{d}\}$. $\square$

4.4 Converging Subseries

In this section we will discuss the three converging subseries numbers $\mathfrak{ß}_c$, $\mathfrak{ß}_e$ and $\mathfrak{ß}_f$, which correspond to sets of subsets of ω that are sufficient to make any conditionally convergent series converge in a certain way. Of these, we know about $\mathfrak{ß}_c$ and $\mathfrak{ß}_f$ that they are *"interesting"*, in the sense that they can be consistently different from both $\aleph_1$ and $\mathfrak{c}$. For $\mathfrak{ß}_e$, we do not know if it can be different from $\mathfrak{c}$.

We start by showing that the three subseries numbers are uncountable.

Lemma 4.4.1

$\mathfrak{ß}_c \geq \aleph_1$. $\triangleleft$

Proof. Let $\overline{s}$ be a sequence of natural numbers such that every number appears infinitely often. Suppose $\mathcal{A} = \{A_n \mid n \in \omega\} \subseteq [\omega]^\omega_\omega$ is countable, then we will recursively define a $\overline{a} \in \textsc{ccs}$ such that $\sum_{n \in A_i} a_n$ diverges.

At the n-th step of the recursion, let $k \in \omega$ be maximal such that a_k has been defined (and take $k = 0$ if all a_k are still undefined), then take natural numbers $k < y_1 < y_2 < \cdots < y_{2n}$ such that $y_i \in A_{s_n}$ if i is even and $y_i \notin A_{s_n}$ if i is odd. Let $a_m = \frac{(-1)^i}{n}$ for every $m = y_i$ and $a_m = 0$ for any other $m \in (k, y_{2n})$ that is unequal to any y_i. It is clear that $\overline{a}$ wil diverge for every A_i, since infinitely often $\sum_{A_i} a_n$ will contain an interval with terms summing to 1. $\square$

We can prove that $\mathfrak{ß}_c$ and $\mathfrak{ß}_f$ are actually the same cardinal characteristic. Unfortunately the proof that we give below does not have the form of a Tukey connection, thus we cannot conclude that $\mathfrak{ß}_c^\perp = \mathfrak{ß}_f^\perp$ as of yet.

Theorem 4.4.2

$\mathfrak{ß}_c = \mathfrak{ß}_f$. $\triangleleft$

Proof. We know $\mathfrak{ß}_c \leq \mathfrak{ß}_f$ as a result of Lemma 2.2.8, thus we only have to prove that $\mathfrak{ß}_f \leq \mathfrak{ß}_c$.

Let $\mathcal{A} \subseteq [\omega]^\omega_\omega$ be such that for every $\overline{a} \in \textsc{ccs}$ there is some $A \in \mathcal{A}$ for which $\sum_{n \in A} a_n$ converges. For every $n \in \omega$ define the set $\mathcal{A}_n = \{A \triangle \{n\} \mid A \in \mathcal{A}\}$, then $\left|\mathcal{A} \cup \bigcup_{n\in\omega} \mathcal{A}_n\right| = |\mathcal{A}| \cdot \aleph_0 = |\mathcal{A}|$.

We claim that for every $\overline{a} \in \textsc{ccs}$ there is some $A \in \mathcal{A} \cup \bigcup_{n\in\omega} \mathcal{A}_n$ for which $\sum_{n\in A} a_n$ converges to a value different from $\sum_{n\in\omega} a_n$. This is because for every $\overline{a} \in \textsc{ccs}$ there is some $A \in \mathcal{A}$ such that $\sum_{n\in A} a_n$ converges. If it turns out that $\sum_{n\in A} a_n = \sum_{n\in\omega} a_n$, then we choose $k \in \omega$ such that $a_k \neq 0$. It then follows that $\sum_{n \in A \triangle \{k\}} a_n \neq \sum_{n\in\omega} a_n$, and because $A \triangle \{k\} \in \mathcal{A}_k$ we see that $\mathcal{A} \cup \bigcup_{n\in\omega} \mathcal{A}_n$ is a witness for $\mathfrak{ß}_c$. $\square$

We can show that $\mathfrak{ß}_c < \mathfrak{c}$ is consistent. In fact, we have an even stronger result, that $\mathfrak{ß}_c \leq \mathfrak{d}$. Since the proof is done by giving a Tukey connection, we also have $\mathfrak{ß}_c^\perp \geq \mathfrak{b}$, which shows that $\mathfrak{ß}_c^\perp$ is uncountable.

Theorem 4.4.3

$\mathfrak{ß}_c \leq \mathfrak{d}$ and $\mathfrak{ß}_c^\perp \geq \mathfrak{b}$. $\triangleleft$

Proof. We give a Tukey connection $\varphi : \underline{\mathscr{D}} \to \mathscr{S}_c$, with $\underline{\mathscr{D}} = \underline{\mathscr{B}}^\perp = \langle \uparrow({}^\omega\omega), \uparrow({}^\omega\omega), \underline{D}\rangle$ where:

$$\underline{D} = \{\langle f, g\rangle \mid \forall^\infty n(|\mathrm{ran}(f) \cap [g(n), g(n+1))| > 1)\}$$

For a conditionally convergent sequence $\overline{a}$, let $\varphi^-(\overline{a}) = f \in {\uparrow}({}^\omega\omega)$ be a function such that $|a_m| < \frac{1}{n^2}$ for all $m > f(n)$ and such that f is strictly increasing. For $g \in {\uparrow}({}^\omega\omega)$ we define $\varphi^+ : {\uparrow}({}^\omega\omega) \to [\omega]^\omega_\omega$ as $\varphi^+(g) = \{2g(n) \mid n \in \omega\}$. We use $\{2g(n) \mid n \in \omega\}$ instead of $\{g(n) \mid n \in \omega\}$ to make sure $\varphi^+(g)$ is coinfinite.

To see that this is a Tukey connection, let $\overline{a} \in \text{CCS}$ and $g \in {\uparrow}({}^\omega\omega)$, and let $f = \varphi^-(\overline{a})$. If $\langle f, g\rangle \in \overline{D}$, then there is $N \in \omega$ such that for all $n \geq N$ we have some k_n such that $f(k_n), f(k_n + 1) \in [g(n), g(n+1))$. It follows that for all $m \in \omega$ with $m > g(n)$ we have $|a_m| < \frac{1}{{k_n}^2}$. It is easy to see that $g(N+n) > f(k_N + n)$, and thus it follows that $|a_{2g(N+n)}| < \frac{1}{(k_N+n)^2}$ for all $n \in \omega$.

From this we see that $\sum_{n\in\omega} a_{2g(N+n)}$ is a convergent series; it is even absolutely convergent. Therefore $\sum_{n\in\varphi^+(g)} a_n$ is convergent, which was needed to show that φ is a Tukey connection. $\square$

Next, we will show that $\mathfrak{s}_c$ can be consistently larger than non(meagre). The model that we use is the Cohen model.

Theorem 4.4.4

$\mathfrak{s}_c = \aleph_2$ in the Cohen model. $\triangleleft$

Proof. Let $\mathbb{P}$ be the forcing with conditions $(s, k) \in {}^{<\omega}\mathbb{Q} \times \omega$, where s is a finite sequence of rationals that will approximate a conditionally convergent sequence. We will abbreviate the domain of the finite sequence as $n_s = \operatorname{dom}(s)$ for the remainder of this proof. We define an ordering as $(t, l) \leq (s, k)$ if $t \supseteq s$, $l \geq k$ and

- $\left|\sum_{n\in[n_s,m)} t(n)\right| \leq \frac{1}{k}$ for every $m < n_t$,
- $\left|\sum_{n\in[n_s,n_t)} t(n)\right| \leq \frac{1}{k} - \frac{1}{l}$.

This order is transitive: let $(t, l) \leq (s, k) \leq (r, j)$, and let $n_s < m < n_t$, then we see that

$$\left|\sum_{n\in[n_r,m)} t(n)\right| \leq \left|\sum_{n\in[n_r,n_s)} s(n)\right| + \left|\sum_{n\in[n_s,m)} t(n)\right| \leq \tfrac{1}{j} - \tfrac{1}{k} + \tfrac{1}{k} = \tfrac{1}{j}$$

and we also see that

$$\begin{aligned}\left|\sum_{n\in[n_r,n_t)} t(n)\right| &\leq \left|\sum_{n\in[n_r,n_s)} s(n)\right| + \left|\sum_{n\in[n_s,n_t)} t(n)\right| \\ &\leq \tfrac{1}{j} - \tfrac{1}{k} + \tfrac{1}{k} - \tfrac{1}{l} = \tfrac{1}{j} - \tfrac{1}{l}.\end{aligned}$$

If G is generic for $\mathbb{P}$, then $\overline{a} = \bigcup\{s \mid (s,k) \in \mathbb{P}\}$ is a conditionally convergent sequence for which $\sum_{n\in A} a_n$ diverges for any $A \in [\omega]^\omega_\omega$ from the ground model. This follows from the following observations.

For any $(s, k) \in \mathbb{P}$ and $m \in \omega$ with $m > k$ there is some $(t, m) \leq (s, k)$ such that

$m \in n_t$ and such that $\sum_{n \in n_t} |t(n)| > m$. We take some $M > mk$ such that $M \setminus n_s$ is even, and let $\mathrm{dom}(t) = M$ with t defined such that $t(n) = \frac{1}{k}$ for all even $n \in M \setminus n_s$ and $t(n) = \frac{-1}{k}$ for all odd $n \in M \setminus n_s$. We therefore see that we can make s as long as we want by increasing its domain, we can make all extensions of s have partial sums that stay within an arbitrarily small interval and we can make the sum of the absolute terms of s as large as we want.

If $A \in [\omega]^\omega_\omega$ and $(s, k) \in \mathbb{P}$, then we can find $(t, k) \leq (s, k)$ with $\sum_{n \in [n_s, n_t) \cap A} t(n) = 1$. To do this, pick n_t such that there are $n_s \leq y_1 < y_2 < \cdots < y_{2k} < n_t$ for which $y_i \in A$ if i is even and $y_i \notin A$ if i is odd. We define $t(n) = \frac{(-1)^i}{k}$ if $n = y_i$ and $t(n) = 0$ if $n \in [n_s, n_t)$ is unequal to all y_i.

As a consequence, we could repeatedly do this, to see that for any $m \in \omega$ there are at least m disjoint intervals I on which $\sum_{n \in A \cap I} t(n) = 1$. From this it follows that the generic conditionally convergent sequence $\overline{a}$ will have infinitely many disjoint intervals I on which $\sum_{n \in A \cap I} a_n = 1$, and thus $\sum_{n \in A} a_n$ diverges.

As $\mathbb{P} = {}^{<\omega}\mathbb{Q} \times \omega$ is countable, by Theorem 1.3.20 we see that $\mathbb{P}$ is forcing equivalent to $\mathbb{C}(\aleph_0)$. Therefore, if we do an iteration of $\mathbb{P}$ of length ω_2 over a model $\mathcal{M} \vDash \mathsf{GCH}$, we will end up with the Cohen model. If $\mathcal{A} \subseteq [\omega]^\omega_\omega \cap \mathcal{M}[G]$ has cardinality $\aleph_1$, then $\mathcal{A}$ will be present in some initial segment of the iteration. By the above argument, in the subsequent step we will add a conditionally convergent sequence $\overline{a}$ for which $\sum_{n \in A} a_n$ diverges for every $A \in \mathcal{A}$. Therefore $\mathcal{A}$ does not satisfy the requirements for $\mathfrak{ß}_c$, thus $\mathfrak{ß}_c = \aleph_2$ in the Cohen model. □

We see that $\mathfrak{ß}_c > \mathrm{non}(\mathsf{meagre})$ is consistent, since in the Cohen model $\mathrm{non}(\mathsf{meagre}) = \aleph_1$.

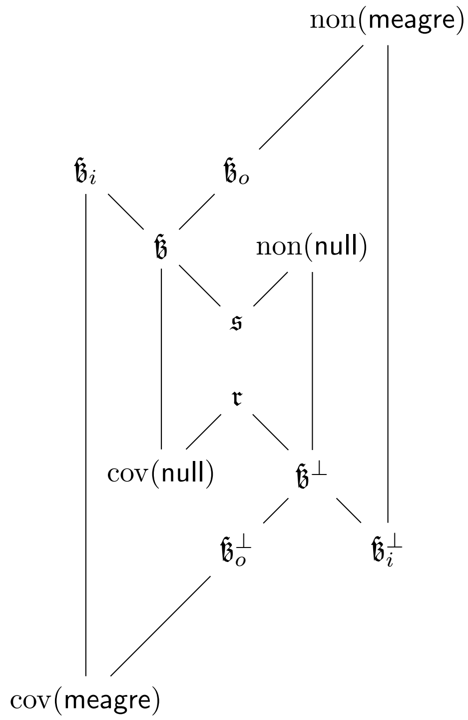
Actually the above theorem gives us a little more than just the consistency of $\mathfrak{ß}_c > \aleph_1$. Due to some absoluteness results by Jindřich Zapletal (see Chapter 6 in [35]), for many *tame* cardinal characteristics $\mathfrak{x}$ and under assumption of a large cardinal it is the case that $\mathrm{cov}(\mathsf{meagre}) > \mathfrak{x}$ is consistent if and only if $\mathfrak{x} = \aleph_1$ in the Cohen model.

One sufficient condition for a cardinal $\mathfrak{x}$ to be *tame* if it is expressable as the least cardinality of a set of reals A such that $\forall x \in {}^\omega\omega \exists y \in A(\theta(x, y))$, where θ is a formula in which all quantifiers are bound on ω or ${}^\omega\omega$ and in which A is not free. Since each conditionally convergent series can be described as a real, as can every infinite coinfinite subset of ω, we can show that $\mathfrak{ß}_c$ is a *tame* cardinal characteristic.

In many cases it turns out that the large cardinal assumption in Zapletal's result is redundant, and that there is a proof in ZFC that $\mathfrak{x}$ is small in the Cohen model if and only if it is consistent that $\mathrm{cov}(\mathsf{meagre}) > \mathfrak{x}$. From such a result it would follow from the above theorem as a corollary that $\mathrm{cov}(\mathsf{meagre}) \leq \mathfrak{ß}_c$ is provable. Unfortunately due to time constraints such an analysis can not be given in this thesis.

4.5 Consistency of Strict Inequalities

In the following diagram we give an overview of some of the results we proved in the previous sections. We leave out the results about combining the subseries numbers with the dominating and bounding numbers and the convergent subseries numbers in order not to clutter up the diagram too much.



As for strict relations, we know that $\mathfrak{ß} > \mathfrak{s}$ and $\mathfrak{ß} > \text{cov}(\mathsf{null})$ are consistent, because it is consistent that $\mathfrak{s} > \text{cov}(\mathsf{null})$ (in the Blass-Shelah model) and that $\text{cov}(\mathsf{null}) > \mathfrak{s}$ (in the random model). Similarly for the dual $\mathfrak{ß}^\perp$ we have that both $\mathfrak{ß}^\perp < \mathfrak{r}$ and $\mathfrak{ß}^\perp < \text{non}(\mathsf{null})$ are consistent, since it is consistent that $\mathfrak{r} < \text{non}(\mathsf{null})$ (in the Blass-Shelah model) and that $\text{non}(\mathsf{null}) < \mathfrak{r}$ (in the random model). This simultaneously shows that all subseries numbers can be consistently larger than all dual subseries numbers.

For the consistency of $\mathfrak{ß}_i > \text{cov}(\mathsf{meagre})$ and $\mathfrak{ß}_i^\perp < \text{non}(\mathsf{meagre})$, we could see that these are the case since it is consistent that $\text{cov}(\mathsf{meagre}) > \text{non}(\mathsf{meagre})$ (in the Cohen model) and that $\text{non}(\mathsf{meagre}) < \text{cov}(\mathsf{meagre})$ (in the random model). This also shows that $\mathfrak{ß}_i, \mathfrak{ß}^\perp$ and $\mathfrak{ß}_o^\perp$ can be consistently larger than $\mathfrak{ß}, \mathfrak{ß}_o$ and $\mathfrak{ß}_i^\perp$.

We have that $\mathfrak{ß}, \mathfrak{ß}_o < \mathfrak{d}$ and $\mathfrak{ß}^\perp, \mathfrak{ß}_o^\perp > \mathfrak{b}$ are consistent by $\text{non}(\mathsf{meagre}) = \mathfrak{b} < \text{cov}(\mathsf{meagre}) = \mathfrak{d}$ being consistent (in the Cohen model), and we have that $\mathfrak{ß}, \mathfrak{ß}_i, \mathfrak{ß}_o > \mathfrak{d}$ and $\mathfrak{ß}^\perp, \mathfrak{ß}_i^\perp, \mathfrak{ß}_o^\perp < \mathfrak{b}$ are consistent by the consistency of $\text{cov}(\mathsf{null}) > \mathfrak{d}$ (in the random model) and $\text{non}(\mathsf{null}) < \mathfrak{b}$ (in the dual random model). In the subseries paper [7] it is proved that $\mathfrak{ß}_o = \aleph_1$ in the Laver model, which also has $\mathfrak{b} = \aleph_2$, thus $\mathfrak{ß}, \mathfrak{ß}_o < \mathfrak{b}$ is consistent as well. The consistency of $\mathfrak{ß}_o^\perp > \mathfrak{d}$ is unknown, as the Laver model can not be dualised in the same manner as we did in Section 3.4. The Laver model also gives us a model where $\mathfrak{F}_o < \mathfrak{rr}$. On the other hand, we do not know that $\mathfrak{F} > \mathfrak{rr}$ is consistent, since non(meagre) $\geq \mathfrak{rr}$, and we do not know if $\mathfrak{rr} <$ non(meagre) is consistent. In case $\mathfrak{rr} =$ non(meagre) turns out to be true, we see that $\mathfrak{F} > \mathfrak{rr}$ is impossible.

As for the converging subseries numbers, we can see that $\mathfrak{F}_c < \mathfrak{F}, \mathfrak{F}_o, \mathfrak{F}_i$ is consistent, since $\mathfrak{F}_c \leq \mathfrak{d} < \mathfrak{F}$ holds in the random model. Dually we have that $\mathfrak{F}_c^\perp > \mathfrak{F}^\perp, \mathfrak{F}_o^\perp, \mathfrak{F}_i^\perp$ is consistent by $\mathfrak{F}_c^\perp \geq \mathfrak{b} > \mathfrak{F}^\perp$ being true in the dual random model. We saw in the previous section that non(meagre) $< \mathfrak{F}_c$ is true in the Cohen model, and thus we see that $\mathfrak{F}_c > \mathfrak{F}_o, \mathfrak{F}$ is consistent as well.

There is no upper bounds are known for $\mathfrak{F}_i$ and $\mathfrak{F}_e$, and we do not know if $\mathfrak{F}_i < \mathfrak{c}$ or $\mathfrak{F}_e < \mathfrak{c}$ and $\mathfrak{F}_i^\perp > \aleph_1$ or $\mathfrak{F}_e^\perp > \aleph_1$ are consistent either.

Conclusion

In this thesis we saw that many of the results from the rearrangement [4] and subseries [7] paper can be formulated in terms of Tukey connections. This gave us the ability to prove many dual results about the dual rearrangement and subseries numbers.

For two results we needed an original method to prove their dual statement. For the proof that $\mathfrak{rr}_o^\perp = \mathfrak{rr}^\perp$ (Theorem 3.3.5) we needed a sequential composition with a relational system for the bounding number $\mathfrak{b}$ to get a satisfactory result. We saw that a similar problem arose in the proof of $\min\left\{\mathfrak{d}, \mathfrak{ß}^\perp\right\} \leq \mathfrak{ß}_o^\perp$ (Theorem 4.3.7), although the original proof that $\mathfrak{ß}_o \leq \max\{\mathfrak{b}, \mathfrak{ß}\}$ from the paper already (implicitly) used a sequential composition, so the proof that we gave is based on the same idea.

We also gave a new way to look at the subseries numbers, by restricting our attention to only the coinfinite subsets of indices of series. In this manner it became reasonable to define a subseries numbers for convergent behaviour, and we saw that the resulting cardinal characteristic $\mathfrak{ß}_c = \mathfrak{ß}_f$ can be consistently larger than $\aleph_1$ and smaller than $\mathfrak{c}$.

There are still many open problems about rearrangement and subseries numbers. To finish this thesis we will give a list with some statements of which the consistency is unknown.

- $\mathfrak{rr} < \mathrm{non}(\mathsf{meagre})$
- $\mathfrak{rr} > \mathfrak{r}$
- $\mathfrak{rr} < \mathfrak{s}$
- $\mathfrak{rr}_{fi} < \mathfrak{rr}_f$
- $\mathfrak{rr}_{fi} < \mathfrak{rr}_i$
- $\mathfrak{rr}_f \neq \mathfrak{rr}_i$
- $\mathfrak{ß}_o > \mathfrak{ß}$
- $\mathfrak{ß}_i < \mathfrak{c}$
- $\mathfrak{ß}_o^\perp > \mathfrak{d}$
- $\mathfrak{ß}_c < \mathrm{cov}(\mathsf{meagre})$
- $\mathfrak{ß}_c^\perp > \mathfrak{ß}_f^\perp$
- $\mathfrak{ß}_e < \mathfrak{c}$

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Overview of Relational Systems

Triple	Description	Norm	Page
$\mathscr{B}$	$\langle {}^\omega\omega, {}^\omega\omega, \not\geq^* \rangle$	$\mathfrak{b}$	28
$\underline{\mathscr{B}}$	$\langle \uparrow({}^\omega\omega), \uparrow({}^\omega\omega), \underline{B} \rangle$ where $\underline{B} = \{\langle f, g\rangle \mid \exists^\infty n(\lvert\operatorname{ran}(g) \cap [f(n), f(n+1))\rvert \leq 1)\}$	$\mathfrak{b}$	31
$\overline{\mathscr{B}}$	$\left\langle [\omega]^\omega_\omega, [\omega]^\omega_\omega, \overline{B} \right\rangle$ where $\overline{B} = \{\langle X, Y\rangle \mid \exists^\infty x, y \in X(x < y \wedge Y \cap [x, y) = \varnothing)\}$	$\mathfrak{b}$	33
$\mathscr{J}$	$\langle [\omega]^\omega, \mathcal{S}(\omega), J \rangle$ where $J = \{\langle X, \pi\rangle \mid \exists^\infty x, y \in X(x < y \wedge \pi(x) > \pi(y))\}$	$\mathfrak{b}$	31
$\mathscr{D}$	$\langle {}^\omega\omega, {}^\omega\omega, \leq^* \rangle$	$\mathfrak{d}$	28
$\underline{\mathscr{D}}$	$\langle \uparrow({}^\omega\omega), \uparrow({}^\omega\omega), \underline{D} \rangle$ where $\underline{D} = \{\langle f, g\rangle \mid \forall^\infty n(\lvert\operatorname{ran}(f) \cap [g(n), g(n+1))\rvert > 1)\}$	$\mathfrak{d}$	70
$\overline{\mathscr{D}}$	$\left\langle [\omega]^\omega_\omega, [\omega]^\omega_\omega, \overline{D} \right\rangle$ where $\overline{D} = \{\langle X, Y\rangle \mid \forall^\infty x, y \in Y(x < y \rightarrow X \cap [x, y) \neq \varnothing)\}$	$\mathfrak{d}$	66
$\mathscr{S}$	$\langle [\omega]^\omega, [\omega]^\omega, \text{is split by} \rangle$	$\mathfrak{s}$	28
$\underline{\mathscr{S}}$	$\langle [\omega]^\omega_\omega, [\omega]^\omega_\omega, \text{is split by} \rangle$	$\mathfrak{s}$	34
$\mathscr{R}$	$\langle [\omega]^\omega, [\omega]^\omega, \text{does not split} \rangle$	$\mathfrak{r}$	28
$\mathscr{C}_I$	$\langle \mathbb{R}, I, \in \rangle$; $\langle {}^\omega\omega, I, \in \rangle$; $\langle {}^\omega 2, I, \in \rangle$; etc.	$\operatorname{cov}(I)$	28
$\mathscr{N}_I$	$\langle I, \mathbb{R}, \not\ni \rangle$; $\langle I, {}^\omega\omega, \not\ni \rangle$; $\langle I, {}^\omega 2, \not\ni \rangle$; etc.	$\operatorname{non}(I)$	28
$\mathscr{R}_o$	$\langle \textsc{ccs}, \mathcal{S}(\omega), R_o \rangle$ where $R_o = \left\{\overline{a}, \pi \;\middle|\; \sum_{n\in\omega} a_{\pi(n)} \text{ diverges by oscillation}\right\}$	$\mathfrak{rr}_o$	41
$\mathscr{R}_i$	$\langle \textsc{ccs}, \mathcal{S}(\omega), R_i \rangle$ where $R_i = \left\{\overline{a}, \pi \;\middle|\; \sum_{n\in\omega} a_{\pi(n)} \text{ diverges to infinity}\right\}$	$\mathfrak{rr}_i$	41
$\mathscr{R}_f$	$\langle \textsc{ccs}, \mathcal{S}(\omega), R_f \rangle$ where $R_f = \left\{\overline{a}, \pi \;\middle|\; \sum_{n\in\omega} a_{\pi(n)} \text{ converges to a new finite limit}\right\}$	$\mathfrak{rr}_f$	41
$\mathscr{R}_{io}$	$\langle \textsc{ccs}, \mathcal{S}(\omega), R_i \cup R_o \rangle$	$\mathfrak{rr}_{io}$	41
$\mathscr{R}_{fi}$	$\langle \textsc{ccs}, \mathcal{S}(\omega), R_f \cup R_i \rangle$	$\mathfrak{rr}_{fi}$	41
$\mathscr{R}_{fo}$	$\langle \textsc{ccs}, \mathcal{S}(\omega), R_f \cup R_o \rangle$	$\mathfrak{rr}_{fo}$	41
$\mathscr{R}_{fio}$	$\langle \textsc{ccs}, \mathcal{S}(\omega), R_f \cup R_i \cup R_o \rangle$	$\mathfrak{rr}$	41

Triple	Description	Norm	Page
$\mathscr{S}_o$	$\langle \textsc{ccs}, [\omega]^\omega, S_o\rangle$ where $S_o = \left\{\overline{a}, A \mid \sum_{n\in A} a_n \text{ diverges by oscillation}\right\}$	$\mathfrak{s}_o$	60
$\mathscr{S}_o'$	$\langle \textsc{ccs}, [\omega]^\omega_\omega, S_o \cap ([\omega]^\omega_\omega \times [\omega]^\omega_\omega)\rangle$	$\mathfrak{s}_o$	60
$\mathscr{S}_i$	$\langle \textsc{ccs}, [\omega]^\omega, S_i\rangle$ where $S_i = \left\{\overline{a}, A \mid \sum_{n\in A} a_n \text{ diverges to infinity}\right\}$	$\mathfrak{s}_i$	60
$\mathscr{S}_i'$	$\langle \textsc{ccs}, [\omega]^\omega_\omega, S_i \cap ([\omega]^\omega_\omega \times [\omega]^\omega_\omega)\rangle$	$\mathfrak{s}_i$	60
$\mathscr{S}_{io}$	$\langle \textsc{ccs}, [\omega]^\omega, S_i \cup S_o\rangle$	$\mathfrak{s}$	60
$\mathscr{S}_{io}'$	$\langle \textsc{ccs}, [\omega]^\omega_\omega, (S_i \cup S_o) \cap ([\omega]^\omega_\omega \times [\omega]^\omega_\omega)\rangle$	$\mathfrak{s}$	60
$\mathscr{S}_c$	$\langle \textsc{ccs}, [\omega]^\omega_\omega, S_c\rangle$ where $S_c = \left\{\langle\overline{a}, A\rangle \mid \sum_{n\in A} a_n \text{ converges}\right\}$	$\mathfrak{s}_c$	61
$\mathscr{S}_f$	$\langle \textsc{ccs}, [\omega]^\omega_\omega, S_f\rangle$ where $S_f = \left\{\langle\overline{a}, A\rangle \mid \sum_{n\in A} a_n \text{ converges to a different finite limit}\right\}$	$\mathfrak{s}_f$	61
$\mathscr{S}_e$	$\langle \textsc{ccs}, [\omega]^\omega_\omega, S_e\rangle$ where $S_e = \left\{\langle\overline{a}, A\rangle \mid \sum_{n\in A} a_n = \sum_{n\in\omega} a_n\right\}$	$\mathfrak{s}_e$	61

List of Symbols

ZFC	Zermelo-Fraenkel set theory with Axiom of Choice	5
AC	Axiom of Choice	5
ZF	Zermelo-Fraenkel set theory without Axiom of Choice	5
$x \setminus y$	relative complement	5
$x \triangle y$	symmetric difference	5
Ord	class of ordinals	5
$\mathrm{cf}(\alpha)$	cofinality of an ordinal α	5
$\mathcal{P}(x)$	power set of x	5
$[x]^\kappa$	set of subsets of x of cardinality κ	5
$[x]^{<\kappa}$	set of subsets of x of cardinality less than κ	5
$\prod_{i \in I} X_i$	cartesian product	6
${}^I X$	set of functions $I \to X$	6
R^{-1}	inverse relation	6
R^c	complementary relation	6
$f[a]$	image of a function f over the set $a \subseteq \mathrm{dom}(f)$	6
$f \restriction a$	restriction of a function f to a set $a \subseteq \mathrm{dom}(f)$	6
$g : X \nrightarrow Y$	partial function from X to Y	6
$\mathrm{Fn}_\kappa(X, Y)$	set of partial functions $X \nrightarrow Y$ of cardinality less than κ	6
${}^{<\alpha}x$	set of sequences of length less than α	6
$\uparrow({}^\alpha\beta)$	set of strictly increasing functions $\alpha \to \beta$	6
$\forall^\infty x \in X$	for all but finitely many $x \in X$	6
$\exists^\infty x \in X$	there exist infinitely many $x \in X$	6
$=^*$	equal except for finitely many elements	6
$\subseteq^*$	subset except for finitely many elements	6
$[\omega]^\omega$	set of infinite subsets of ω	8
$[\omega]^\omega_\omega$	set of infinite coinfinite subsets of ω	8
$\mathcal{S}(\omega)$	set of permutations on ω	8
$\mathfrak{c}$	cardinality of the continuum, $2^{\aleph_0}$	8
CH	continuum hypothesis	8
GCH	generalised continuum hypothesis	8
null	σ-ideal of Lebesgue null sets of the continuum	9

Index