Higher Baire Spaces

Cardinal Characteristics, Higher Reals & Bounded Spaces

Dissertation

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Dedicated to the memory of Cees van der Straaten, †2022

Ik zou wiskunde nooit zo leuk hebben kunnen vinden, als ik niet zulk een enthousiaste, geestige en stimulerende eerste wiskundeleraar had gehad.

Contents

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Remark on Authorship

I declare that the results presented in this dissertation are, unless stated otherwise, solely my own work. Results that can be attributed to others in the literature will be attributed as such, either to the primary source, to a standard reference or as "folklore" if no primary source is known by me. Critical ideas that were contributed by someone other than the author will be marked as such, either in the text directly preceding the result, or in a footnote. Particularly, many results in this dissertation are generalisations of known results from the context of the Baire space to the context of higher Baire spaces. In those cases, the original result will be mentioned as "cf. [source] for ${}^\omega\omega$".

Related Publications

Parts of this dissertation have been published or submitted for publication.

- [vdV23] *"Cardinal characteristics on bounded generalised Baire spaces"* is a preprint authored by me that has been submitted for publication in October 2023 and is available on arXiv.
  `https://arxiv.org/abs/2307.14118`
  The contents of this paper coincide with the results presented in Chapters 3 and 6 and parts of Chapter 4 (particularly those results related to bounded higher Baire spaces).
- [vdV] *"Separating many localisation cardinals on generalised Baire spaces"* is a paper authored by me that was published online in the Journal of Symbolic Logic in April 2023.
  `https://doi.org/10.1017/jsl.2023.21`
  The contents of this paper coincide with the results presented in Chapter 5.

1

Introduction

A synopsis of the main new results presented in this dissertation can be found in both English and German in Appendices A.1 and A.2.

Although the name would imply that set theorists mostly study specific sets, in practice it is more accurate to say that set theorists study the independence of mathematical statements relative to a broad family of logical theories that have a primitive notion of "being an element of something". The most commonly known and studied such theory is ZFC, or Zermelo-Fraenkel Set Theory with the Axiom of Choice. This theory is of such generality that it can serve as an axiomatic basis for the vast majority of mathematical literature.

By Gödel's Incompleteness Theorems, it is nevertheless impossible for a sufficiently strong theory to be an axiomatic basis that decides the truth of *all* mathematical statements, and ZFC is indeed no exception. Such unprovable statements are called *independent* of ZFC, and not infrequently examples of independent statements appear in fields of research outside of set theory.

One particularly rich source of independent statements, is the study of the real line (also known as the continuum) with the most famous example being the *Continuum Hypothesis* (CH), or, the statement that every infinite subset of the real numbers is either countable or has the same cardinality as the set of all real numbers. Paul Cohen introduced the method of forcing in 1963 [Coh63], which is effectively a method using a partial ordered set (called a *forcing notion*) to convert a model of ZFC (called the *ground model*) into a new model of ZFC (called a *forcing extension*). A (nontrivial) forcing extension contains many new sets that are not found in the ground model, and may, for instance, contain many new real numbers. By careful control of the properties of the forcing notion, Cohen was able to show that any ground model has a forcing extension in which CH holds[1], and a forcing extension in which CH fails.

Under the assumption that CH fails, we see that there are sets of real numbers that are uncountable, but of strictly smaller cardinality than the real numbers. An interesting question is therefore if there exist such sets with interesting mathematical properties, and if so, what possible cardinality these sets can have. Cardinalities like this are known as *cardinal characteristics of the continuum*, and examples include the least cardinality of a set of positive Lebesgue outer measure, the least number of (closed) nowhere dense sets necessary to cover the real line, or the least cardinality of a base for a nonprincipal ultrafilter on the natural numbers[2]. For each of

[1] The consistency of CH had already been proved 25 years earlier, when Gödel announced his constructible universe in the two-page paper [Gö38].

[2] On first inspection, the latter two examples do not appear to be related to sets of real numbers. However, on second inspection, one could note that both closed nowhere dense sets and subsets of the natural numbers can be *coded* by real numbers.

these examples, the method of forcing can be used to show they can consistently (relative to ZFC) attain a wide variety of cardinalities different from the continuum.

Set theorists usually work with the Cantor space ${}^{\omega}2$ (of binary sequences indexed by the natural numbers ω) and the Baire space ${}^{\omega}\omega$ (of functions from ω to ω) instead of the real numbers, as this change of space generally does not affect the size of cardinal characteristics. In the most recent three decades, an increasing number of people have been studying what happens if we replace the natural numbers ω with a set of uncountable cardinality. A *higher Baire space* ${}^{\kappa}\kappa$, also known as *generalised Baire space*[3], where κ is an uncountable cardinal number, allows us to define cardinal characteristics of higher Baire spaces. For example, one could define a (natural) topology on ${}^{\kappa}\kappa$ and consider the least number of nowhere dense sets necessary to cover ${}^{\kappa}\kappa$.

The overarching theme of this thesis is **cardinal characteristics of higher Baire spaces**, with a special focus on cardinal characteristics that are classically (i.e. in the context of ${}^{\omega}\omega$) associated with the *Cichoń diagram*[4]. The Cichoń diagram shows the relationship between ten cardinal characteristics, defined in terms of the ideal of Lebesgue null sets, the ideal of meagre sets (that is, sets that are countable unions of nowhere dense sets) and dominating and unbounded subsets of ${}^{\omega}\omega$ (where $D \subseteq {}^{\omega}\omega$ is *dominating* if every function $f \in {}^{\omega}\omega$ is eventually bound by some function $g \in D$ in the sense that $f(n) \leq g(n)$ for all n that are large enough; and $B \subseteq {}^{\omega}\omega$ is *unbounded* if no function $f \in {}^{\omega}\omega$ eventually bounds all elements of B).

Some concepts readily generalise from ${}^{\omega}\omega$ to ${}^{\kappa}\kappa$ (such as meagre, dominating and unbounded subsets of ${}^{\omega}\omega$), whereas others prove to be quite difficult to generalise (such as Lebesgue measure and the associated null ideal). On the other hand, higher Baire spaces allow for certain structures that have no classical analogue as well (such as the existence of limit ordinals below κ and stationary sets).

Mathematical results regarding cardinal characteristics can be broadly categorised into ZFC-results on the one hand and proofs of independence on the other hand. A ZFC-result shows a relation between cardinal characteristics that is provable in ZFC. Consequently, such statements have the same truth value in any model of ZFC and thus their truth is preserved under the method of forcing. Contrastingly, proofs of independence show that two (or more) cardinal characteristics are consistently different from each other, and usually involve the method of forcing to produce a model that exhibits this difference. In order to show independence, careful, and frequently quite technical, analysis of forcing notions is necessary to show that the forcing extension has the right properties.

In some cases, independence of ZFC can only be shown relative to an axiomatic system that is strictly stronger than ZFC. Many such stronger systems are the product of assuming that

[3] We chose to prefer the term "higher" over "generalised", for reasons explained in the introduction of [BGS20], namely that "higher" follows analogy with other terms, e.g. higher Suslin trees, higher recursion theory, etc.

[4] Named by Fremlin [Fre84] after Jacek Cichoń. Although the ideas for the proof behind the relations in the Cichoń diagram were discovered by other people (Rothberger, Miller, Raisonnier, Stern, Bartoszyński), the paper [CKP85] written by Cichoń, Kamburelis and Pawlikowski is one of the earlier papers considering previously known statements about null and meagre sets and dominating and unbounded families in terms of cardinal numbers.

there exist uncountable cardinals with certain combinatorial properties, called ***large cardinals***. Although large cardinal assumptions are not frequently needed in the study of classical cardinal characteristics, they play a vital role in the study of higher cardinal characteristics.

In this dissertation, we will give an overview of cardinal characteristics that generalise the Cichoń diagram to higher Baire spaces ${}^\kappa\kappa$, with a special focus on cases where κ has the (large cardinal) property of being inaccessible. A significant part of the dissertation is concerned with variants of the cardinal characteristics of the higher Cichoń diagram that are obtained by limiting the set of functions in ${}^\kappa\kappa$ to only those that are bound below some fixed function $b \in {}^\kappa\kappa$. These bounded variants have been studied in the context of ${}^\omega\omega$ before, but are first studied on ${}^\kappa\kappa$ in this dissertation, and we will also introduce a couple of bounded variants that do not have any analogue in ${}^\omega\omega$, as far as we are aware.

We will both show ZFC-results regarding the relative order of cardinal characteristics and the influence on the choice of bound b on the variants that are defined on bounded higher Baire spaces. We will also study the effect that several forcing notions have on the size of the cardinal characteristics, which will lead us to new independence results.

Although this introduction has been written for a broader audience, the rest of this dissertation assumes the reader is familiar with set theory. In particular, we assume familiarity with the theory of forcing as treated in e.g. [Jec86, Kun11, Hal11], and will not define the method of forcing in this dissertation. We also assume the reader is familiar with some well-known large cardinals (inaccessible, weakly compact, measurable).

1.1. Structure of the Dissertation

We will first establish the general mathematical notation and conventions used in this dissertation in Section 1.2.

Chapter 2 will form the background to the rest of the dissertation and contains no new results by the author. We will formally define higher Baire spaces and discuss their properties. We will furthermore define our main cardinal characteristics using the framework of relational systems. Relational systems will help us in giving concise ZFC-results regarding our cardinal characteristics. We will define both the cardinal characteristics of the classical and higher Cichoń diagrams and give some (sketches of) proofs of the relations between these cardinal characteristics, and an overview of unknown relations.

In Chapter 3 we introduce bounded higher Baire spaces, as well as bounded versions of the cardinal characteristics of the higher Cichoń diagram. We will prove ZFC-results regarding the relation between the bounded and unbounded cardinal characteristics of the higher Cichoń diagram. Finally, we discuss the influence of the choice of bound (and of other parameters) on these cardinals, in particular for which choices the cardinals do not consistently lie strictly between κ and κ^+. This leads to several interesting open questions as well.

In Chapter 4 we will discuss forcing notions associated with higher Baire spaces. We give properties of such forcing notions and how these properties will influence our cardinal characteristics.

We do this by investigating new elements of ${}^{\kappa}\kappa$ with certain generic combinatorial properties over the ground model. We will also give some simple independence results, most of them old, some of them new.

The last two chapters deal with more complex independence results. In Chapter 5 we show the consistency of a large family (of size κ^+) of localisation cardinals with distinct cardinalities. In Chapter 6 we show the existence of a family (of size κ) of cardinals, any finite subset of which yields a forcing extension where all of the cardinals in the finite subset are distinct.

1.2. Notation

Our notation will be mostly standard, following references such as [Jec03, Kun11]. We will use the Greek letters $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \xi$ for ordinals, while $\kappa, \lambda, \mu, \nu$ will be used for (usually infinite) cardinals. The class of all ordinals is written as **Ord**. Blackboard boldface uppercase letters $\mathsf{A}, \mathsf{B}, \mathsf{C}, \cdots$ are reserved for forcing notions, fraktur lowercase letters $\mathsf{a}, \mathsf{b}, \mathsf{c}, \cdots$ are reserved for cardinal characteristics and script uppercase letters $\mathit{A}, \mathit{B}, \mathit{C}, \cdots$ are reserved for relational systems.

We say that a property P holds for ***almost all*** $\alpha \in \kappa$ if there is a $\beta \in \kappa$ such that $P(\alpha)$ holds for all $\alpha \geq \beta$, and we abbreviate this as $\forall^{\infty}\alpha \in \kappa P(\alpha)$. Dually, a property is said to hold for ***cofinally many*** $\alpha \in \kappa$ if for every $\beta \in \kappa$ there is some $\alpha \geq \beta$ for which $P(\alpha)$ holds, abbreviated as $\exists^{\infty}\alpha \in \kappa P(\alpha)$. Given two functions f, f' with domain κ and C a relation defined on $\mathrm{ran}(f) \times \mathrm{ran}(f')$, we write

- $f \mathrel{\mathsf{C}} f'$ as a shorthand for $\forall \alpha \in \kappa(f(\alpha) \mathrel{\mathsf{C}} f'(\alpha))$,
- $f \mathrel{\mathsf{C}^*} f'$ as a shorthand for $\forall^{\infty}\alpha \in \kappa(f(\alpha) \mathrel{\mathsf{C}} f'(\alpha))$,
- $f \mathrel{\mathsf{C}^{\infty}} f'$ as a shorthand for $\exists^{\infty}\alpha \in \kappa(f(\alpha) \mathrel{\mathsf{C}} f'(\alpha))$.

The intended meaning of C^* and C^{∞} are the negations of C^* and C^{∞} respectively, as should be clear on sight, in contrast to the ambiguously notated $\not\mathsf{C}^*$ and $\not\mathsf{C}^{\infty}$. For this reason we will henceforth use the former notation.

Subsets and Functions

Unsurprisingly, we write cardinal exponentiation as λ^{μ} and we let $\lambda^{<\mu} = \bigcup_{\alpha\in\mu} \lambda^{|\alpha|}$. If instead we want to discuss the set of functions from X to Y, we write this as ${}^{X}Y$. If α is an ordinal, then ${}^{<\alpha}Y$ denotes the set $\bigcup_{\xi\in\alpha} {}^{\xi}Y$. We define

$$[Y]^{\mu} = \{X \in \mathcal{P}(Y) \mid |X| = \mu\},$$
$$[Y]^{<\mu} = \{X \in \mathcal{P}(Y) \mid |X| < \mu\}.$$

Naturally, ${}^{\leq\alpha}Y$ and $[Y]^{\leq\alpha}$ have the obvious meaning that is implicit from the above.

We will consider functions with domain α to be the same concept as a sequence of length α. Given sequences $s \in {}^{\alpha}X$ and $t \in {}^{\beta}X$, we write $s^\frown t \in {}^{\alpha+\beta}X$ for the concatenation of s and t.

We write $\langle x \rangle$ for the sequence of length 1 containing only x. If $s \in {}^{\alpha}X$ is a sequence, we write $\mathrm{dom}(s) = \alpha$ for the length, order-type or domain of s. If $A \subseteq \mathbf{Ord}$ is a set of ordinals, we write $\mathrm{ot}(A)$ for the order-type of $\langle A, \in \rangle$. If f is a function and $A \subseteq X$, we write $f \upharpoonright A$ for the restriction of f to A. The range of a function f is denoted as $\mathrm{ran}(f)$, and if $A \subseteq \mathrm{dom}(f)$, we write $f[A]$ for $\mathrm{ran}(f \upharpoonright A)$.

If f, g are functions on ordinals, we interpret arithmetical operators on the functions elementwise, such as $f + g : \alpha \mapsto f(\alpha) + g(\alpha)$ and $2^f : \alpha \mapsto 2^{f(\alpha)}$ and if ξ is an ordinal $f + \xi : \alpha \mapsto f(\alpha) + \xi$. We will often work with functions b where $b(\alpha)$ is a cardinal for each α. In such cases, we also establish that $\mathrm{cf}(b) : \alpha \mapsto \mathrm{cf}(b(\alpha))$ and $b^+ : \alpha \mapsto (b(\alpha))^+$. Finally if α is an ordinal, we write $\overline{\alpha}$ for the constant function $\kappa \to \{\alpha\}$.

An increasing function $f : \kappa \to \mathbf{Ord}$ is called *continuous*[5] at $\gamma \in \kappa$ if $f(\gamma) = \bigcup_{\alpha<\gamma} f(\alpha)$ and otherwise it is called *discontinuous* at $\gamma \in \kappa$. If $A \subseteq \kappa$, then f is (dis)continuous on A if f is (dis)continuous at γ for every limit ordinal $\gamma \in A$.

Trees

Let κ be an infinite cardinal and $\mathcal{F}$ be a set of functions such that $\mathrm{dom}(f) = \kappa$ for each $f \in \mathcal{F}$, then we define the set of *initial segments* $\mathcal{F}_{<\kappa} = \{f \upharpoonright \alpha \mid f \in \mathcal{F} \wedge \alpha \in \kappa\}$ of functions in $\mathcal{F}$.

A subset $T \subseteq \mathcal{F}_{<\kappa}$ is called a *tree on* $\mathcal{F}$ if for every $u \in T$ and $\beta \in \mathrm{dom}(u)$ we have $u \upharpoonright \beta \in T$. A subset $C \subseteq T$ is a *chain* if for any $u, v \in C$ we have $u \subseteq v$ or $v \subseteq u$, and C is called *maximal* if there exists no chain $C' \subseteq T$ with $C \subsetneq C'$. A function $b : \alpha \to \kappa$ where $\alpha \leq \kappa$ is called a *branch* of T if there exists a maximal chain $C \subseteq T$ such that $b = \bigcup C$. The set of branches of T is denoted by $[T]$. We define the *subtree of* T *generated by* $u \in T$ as:

$$(T)_u = \{v \in T \mid u \subseteq v \vee v \subseteq u\}\,.$$

If $u \in T$, let $v \in T$ be a *successor* of u if there exists x such that $v = u^\frown\langle x \rangle$. We denote the set of successors of u in T by $\mathrm{suc}(u, T)$.

We call u a λ*-splitting* node (of T), if $|\mathrm{suc}(u, T)| \geq \lambda$. We say u is a *splitting* node if it is a 2-splitting node, and a *non-splitting* node otherwise. If u is a λ-splitting node, but not a μ-splitting node for any cardinal μ with $\lambda < \mu$, then we say that u is a *sharp* λ-splitting node. We let $\mathrm{Split}_\alpha(T)$ be the set of all $u \in T$ such that u is splitting and

$$\mathrm{ot}(\{\beta \in \mathrm{dom}(u) \mid u \upharpoonright \beta \text{ is splitting}\}) = \alpha.$$

We let $\mathrm{Lev}_\xi(T) = \{u \in T \mid \mathrm{dom}(u) = \xi\}$ denote the ξ-th level of the tree T. We also introduce the following shorthands.

$$\mathrm{Split}_{<\alpha}(T) = \textstyle\bigcup_{\xi<\alpha} \mathrm{Split}_\xi(T) \qquad \mathrm{Lev}_{<\alpha}(T) = \bigcup_{\xi<\alpha} \mathrm{Lev}_\xi(T)$$
$$\mathrm{Split}_{\leq\alpha}(T) = \mathrm{Split}_{<\alpha+1}(T) \qquad \mathrm{Lev}_{\leq\alpha}(T) = \mathrm{Lev}_{<\alpha+1}(T)$$
$$\mathrm{Split}(T) = \mathrm{Split}_{<\kappa}(T)$$

[5] This agrees with the topological notion of continuity under the order topology.

Forcing Conventions and Notation

We force downwards, that is, if P is a forcing notion and $p, q \in \mathsf{P}$ are conditions, then $q \leq_{\mathsf{P}} p$ means that q is a ***stronger*** condition than p, and thus tells us ***more*** about the generic object. The largest or trivial condition of P is denoted as $\mathbb{1}_{\mathsf{P}}$. If two conditions p, q are incompatible, i.e. have no common lower bound, we write $p \perp_{\mathsf{P}} q$, and otherwise we write $p \parallel_{\mathsf{P}} q$. We leave out the subscript P whenever the choice of forcing notion is clear from context, including in the "forces" symbol $\ _{\mathsf{P}}$.

We will denote the ground model with $\mathbf{V}$ and write $\mathbf{V}^{\mathsf{P}}$ for an arbitrary forcing extension by the forcing P. If G is P-generic over $\mathbf{V}$, we write $\mathbf{V}[G]$ for the forcing extension that is specifically given by G. We will write names with a dot, e.g. $\dot{A}, \dot{f}, \dot{\varphi}, \cdots$. Canonical names for objects from the ground model will usually be unmarked, but may be occasionally denoted with a check, e.g. $\check{\alpha}, \check{g}, \check{X}, \cdots$, for the sake of clarity or emphasis.

2

Cardinal Characteristics on Higher Baire Spaces

In this section we give the general background to this dissertation. This section will not contain any new results and can be seen as a concise survey of the (higher) Cichoń diagram.

We will discuss first the classical continuum, including Polish spaces and σ-ideals on the reals in Section 2.1, before we move on to higher Baire spaces in Section 2.2. In order to efficiently talk about cardinal characteristics, we then introduce the machinery of relational systems in Section 2.3 and subsequently in Section 2.4 give definitions for the (classical) cardinal characteristics of the Cichoń diagram and generalisations thereof to higher Baire spaces. We will give an overview of results regarding the higher Cichoń diagram that were known before the writing of this dissertation in Section 2.5. Finally we will conclude the section with an overview of open problems.

2.1. The Continuum

When defining the real numbers, one usually does so by establishing a set of axioms that the reals satisfy (a complete linear ordering that has a countable dense subset isomorphic to the rationals), or construct the reals from the natural numbers, via rational numbers, as the collection of Dedekind cuts or as the collection of limits of Cauchy sequences. Cantor has proved that it does not matter which of these definitions is used, as any linear order satisfying the same axioms as the reals is isomorphic to the reals. We will therefore write $\mathbf{R}$ to denote the reals as described above.

In the set theoretic study of the reals, it is often inconvenient to work with $\mathbf{R}$ directly. Instead, we frequently work with other perfect Polish spaces (i.e. topological spaces that are separable, completely metrisable, and have no isolated points). Of special importance are the ***Cantor space*** ${}^\omega 2$, which can be found as (the isomorphic image of) a closed subset of any other perfect Polish space, or the ***Baire space*** ${}^\omega\omega$, which continuously maps onto any other perfect Polish space.

A topology on both ${}^\omega 2$ and ${}^\omega\omega$ is defined by giving $2 = \{0, 1\}$ and ω the discrete topology, and equipping ${}^\omega 2$ and ${}^\omega\omega$ with the product topology. Equivalently, a basis of open sets is given by sets of the form $[s] = \{f \in {}^\omega\omega \mid s \subseteq f\}$ for initial segments $s \in {}^{<\omega}\omega$ (and similarly for ${}^\omega 2$).[1] Note that $\mathbf{R}$ is almost homeomorphic to ${}^\omega 2$ and ${}^\omega\omega$, in the sense that the sets are homeomorphic after a countable set of exceptions is removed.[2] Removing a countable set is in a certain sense

[1] In fact, such sets $[s]$ are clopen, that is, both closed and open.

[2] Precisely, ${}^\omega\omega$ is homeomorphic to the irrational numbers, and the half-open interval $[0, 1)$ $\mathbf{R}$ is homeomorphic to the subset of ${}^\omega 2$ consisting of those f 2 ${}^\omega 2$ that are not eventually constant with value 1.

negligible in comparison to the (uncountable) size of the continuum, and thus the choice of perfect Polish space usually has no influence on the cardinality of naturally defined uncountable sets of reals. For this reason, it is custom in the set theoretic study of the reals to brand the elements of ***any*** perfect Polish space as ***reals***, a custom we will follow as well.

Let us mention some other perfect Polish spaces. For any countable sequence $\langle A_n \mid n \in \omega \rangle$ of countable sets of size at least 2, we can give each A_n the discrete topology and consider the space $\prod_{n\in\omega} A_n$ with the product topology, then this is a perfect Polish space. By considering characteristic functions, one can identify the power set $\mathcal{P}(\omega)$ with the Cantor space ${}^{\omega}2$. Finally, the intervals $[0,1]$, $(0,1)$, $[0,\infty)$, etc. as subsets of $\mathbf{R}$ are also perfect Polish spaces.[3]

Let P be your favourite perfect Polish space. A (proper) ***σ-ideal*** on P is a nonempty family of sets $\mathcal{I} \subseteq \mathcal{P}(P)$ that is closed under subsets, closed under countable unions and such that $P \notin \mathcal{I}$. We say a subset $X \subseteq P$ is ***$\mathcal{I}$-negligible*** if $X \in \mathcal{I}$, and otherwise call X an ***$\mathcal{I}$-positive*** set. If $P \setminus X \in \mathcal{I}$, we say that X is ***$\mathcal{I}$-full***. The set of $\mathcal{I}$-full subsets of P is a ***σ-filter*** (i.e. closed under countable intersections and supersets) called the ***dual filter***. We are particularly interested in the following three σ-ideals on the reals:

Definition 2.1.1
The ***meagre ideal*** $\mathcal{M} \subseteq \mathcal{P}({}^{\omega}\omega)$ is the ideal of meagre subsets of ${}^{\omega}\omega$. A set $X \subseteq {}^{\omega}\omega$ is called ***nowhere dense***, if any open neighbourhood contains an open neighbourhood disjoint from X. A set $X \subseteq {}^{\omega}\omega$ is called ***meagre*** if it is the countable union of nowhere dense sets. Sets that are $\mathcal{M}$-full are also called ***comeagre***.

The ***null ideal*** $\mathcal{N} \subseteq \mathcal{P}({}^{\omega}2)$ is the ideal of (Lebesgue) null sets. We define a measure on the basic open subsets $[s]$ of ${}^{\omega}2$ by setting $\mu([s]) = 2^{-n}$ where n is such that $s \in {}^{n}2$. This induces a Lebesgue measure on the Borel subsets of ${}^{\omega}2$, and we say $X \subseteq {}^{\omega}2$ is ***null*** if there exists a Borel set $B \subseteq {}^{\omega}2$ with $X \subseteq B$ such that $\mu(B) = 0$. Measurable sets that are $\mathcal{N}$-positive have a positive Lebesgue measure, and sets that are $\mathcal{N}$-full have Lebesgue measure 1.

The ***strong measure zero ideal*** $\mathcal{SN} \subseteq \mathcal{P}({}^{\omega}2)$ is the ideal of strong measure zero sets. A set $X \subseteq {}^{\omega}2$ is called ***strong measure zero*** if for every $f \in {}^{\omega}\omega$ there exists a sequence $\left\langle s_n \in {}^{f(n)}2 \mid n \in \omega \right\rangle$ such that $X \subseteq \bigcup_{n\in\omega}[s_n]$. $\triangleleft$

Under the assumption that the Continuum Hypothesis (CH) fails, there may be many uncountable cardinalities that are strictly smaller than the cardinality of the continuum $\mathfrak{c} = 2^{\aleph_0}$. We may ask the question whether there exist subsets of the reals with interesting properties that are uncountable, yet consistently strictly below $\mathfrak{c}$. ***Cardinal characteristics of the continuum*** are cardinalities between $\aleph_1$ and $\mathfrak{c}$ associated with sets of real numbers, that are consistently different from either bound.[4]

[3] We refer to [Kec95, Chapter 3] for a detailed exposition on Polish spaces and the facts mentioned in this paragraph.

[4] We will take some liberty with the bounds on size in this definition. For instance, a cardinal characteristic known as the *cofinality of the strong measure zero ideal* is consistently strictly *larger* than $\mathfrak{c}$, but we will still consider it a cardinal characteristic of the continuum due to its intimate connection with the reals.

We will introduce a variety of cardinal characteristics of the continuum and their properties after we have introduced higher Baire spaces and relational systems.

2.2. Higher Baire spaces

Let κ be an uncountable cardinal. A ***higher Baire space*** ${}^\kappa\kappa$ is the result of replacing ω in the definition of the classical Baire space ${}^\omega\omega$ by an uncountable cardinal κ, that is, ${}^\kappa\kappa$ is the set of functions $f : \kappa \to \kappa$. By replacing "finite" by "$<\kappa$" and "countable" by "$\leq\kappa$", we may generalise many aspects of ${}^\omega\omega$ to ${}^\kappa\kappa$. A topology on ${}^\kappa\kappa$ is given by the ***$<\kappa$-box topology***, that is, basic opens of ${}^\kappa\kappa$ are sets are of the form $\prod_{\alpha\in\kappa} O_\alpha$, where each $O_\alpha \subseteq \kappa$ and the set of α such that $O_\alpha \neq \kappa$ is smaller than κ. Equivalently, a basis of clopens is given by sets of the form $[s] = \{f \in {}^\kappa\kappa \mid s \subseteq f\}$ for initial segments $s \in {}^{<\kappa}\kappa$. In a very similar manner, we can define ***higher Cantor spaces*** as sets ${}^\kappa 2$, with a topology defined as above. In analogy with elements of ${}^\omega\omega$ being called ***reals***, the elements of ${}^\kappa\kappa$ or ${}^\kappa 2$ will be called ***κ-reals***.

Starting with a paper by Cummings & Shelah [CS95], there have been significant developments in generalising the theory of cardinal characteristics from ${}^\omega\omega$ to ${}^\kappa\kappa$, where κ is an uncountable cardinal. A cardinal characteristic of the higher Baire space ${}^\kappa\kappa$, is a cardinality between κ^+ and 2^κ that is consistently different from either bound.

The above topology on ${}^\kappa\kappa$ lets us define (proper) $\leq\kappa$-complete ideals on ${}^\kappa\kappa$ that are analogous to $\mathcal{M}$ and $\mathcal{SN}$. A (proper) $\leq\kappa$-complete ideal on a space P is a nonempty family of sets $\mathcal{I} \subseteq \mathcal{P}(P)$ that is closed under subsets, closed under unions of size κ and such that $P \notin \mathcal{I}$. We present the following two $\leq\kappa$-complete ideals:

Definition 2.2.1
The ***κ-meagre ideal*** $\mathcal{M}_\kappa \subseteq \mathcal{P}({}^\kappa\kappa)$ is the ideal of κ-meagre subsets of ${}^\kappa\kappa$. A set $X \subseteq {}^\kappa\kappa$ is called ***κ-meagre*** if it is the union of κ many nowhere dense sets.

The ***κ-strong measure zero ideal*** $\mathcal{SN}_\kappa \subseteq \mathcal{P}({}^\kappa 2)$ is the ideal of κ-strong measure zero sets. A set $X \subseteq {}^\kappa 2$ is called ***κ-strong measure zero*** if for every $f \in {}^\kappa\kappa$ there exists a sequence $\langle s_\alpha \in {}^{f(\alpha)}2 \mid \alpha \in \kappa\rangle$ such that $X \subseteq \bigcup_{\alpha\in\kappa}[s_\alpha]$. ◁

In comparison to the three σ-ideals mentioned in Definition 2.1.1, the ideal of Lebesgue null sets is noteworthy for its absence. This is because measurability is not easy to generalise to higher cardinals. The existence of a $<\kappa$-complete measure on κ is well-known to be equivalent to the existence of a measurable cardinal.[5] The situation for Lebesgue measure on ${}^\kappa\kappa$ is worse, as it is unclear how to generalise (infinite) summation of real numbers in a manner that allows us to define Lebesgue measure.[6]

There have been several solutions proposed to be able to talk about the null ideal or its related cardinal characteristics. One method of obtaining a higher null ideal, is by defining a forcing

[5] This was shown by Solovay [Sol71].

[6] See for example Chapter 5 in the PhD dissertation of Wontner [Won23] for a detailed overview of summation of generalisations of the real numbers. There, a list of desiderata for infinite summation is presented and it is proved that there does not exist a generalisation of summation that satisfies all of them.

notion closely resembling random forcing, and using this forcing notion to define an ideal. Shelah [She17] defined an ideal $\mathrm{id}(\mathsf{Q}_\kappa)$ for inaccessible κ from a forcing notion Q_κ with similar properties to random forcing, making $\mathrm{id}(\mathsf{Q}_\kappa)$ resemble the classical null ideal.

A different forcing notion F resembling random forcing has been proposed by Friedman & Laguzzi [FL17]. Unfortunately, the construction of F requires the assumption of a diamond principle[7] that implies $2^\kappa = \kappa^+$. Under the latter, cardinal characteristics of the higher continuum are rather boring, as they can only have the value κ^+.

In this dissertation we take a different approach, and do not generalise the null ideal directly. Instead, we generalise two combinatorial cardinal characteristics of the continuum that can be defined without mention of the null ideal. In the classical case, these two cardinals can be shown to be equivalent to two of the cardinal characteristics related to the null ideal (see Fact 2.4.3).

2.3. Relational Systems

In order to define our cardinal characteristics and study the relations between them, we will make use of relational systems and Tukey connections. Such systems were first defined and studied by Tukey [Tuk40] in the context of coverings and uniformity of topological spaces, and later applied to (other) cardinal characteristics by Fremlin [Fre84] and Vojtáš [Voj93]. All of the cardinal characteristics we are interested in, can be expressed as the norm of a relational system. We will only give a brief overview of relational systems below, and refer to [Bla10, Section 4] for a detailed description.

A relational system $\boldsymbol{R} = \langle X, Y, R\rangle$ is a triple of sets X and Y and a relation $R \subseteq X \times Y$. We define the ***norm*** of $\boldsymbol{R}$ (if it is not undefined) as

$$\|\boldsymbol{R}\| = \min\{|W| \mid W \subseteq Y \text{ and } \forall x \in X \exists y \in W (x\ R\ y)\}.$$

We refer to a set $W \subseteq Y$ such that $\forall x \in X \exists y \in W (x\ R\ y)$ as a ***witness*** for $\|\boldsymbol{R}\|$. We define a ***dual*** relation to R by

$$R^\perp = \{(y, x) \in Y \times X \mid (x, y) \notin R\}.$$

Correspondingly we can define the dual relational system $\boldsymbol{R}^\perp = \left\langle Y, X, R^\perp\right\rangle$.

Intuitively, we can see Y as a set of possible responses to a set of potential challenges X. We want to find a set of responses $Y' \subseteq Y$ such that every challenge can be met with a response from Y', and the norm expresses the least number of responses necessary to do so. The dual relational system answers the question how many challenges we should gather such that no single response can meet them all.

Given two relational systems $\boldsymbol{R} = \langle X, Y, R\rangle$ and $\boldsymbol{R}' = \langle X', Y', R'\rangle$, a ***Tukey connection*** from $\boldsymbol{R}$ to $\boldsymbol{R}'$ is a pair of functions $\rho_- : X \to X'$ and $\rho_+ : Y' \to Y$ such that for any $x \in X$ and $y' \in Y'$

[7] To be precise, of $_{\kappa^+}(S^\kappa_{\kappa^+})$.

with $(\rho_-(x), y') \in R'$ we also have $(x, \rho_+(y')) \in R$. We let $\mathbf{R} \preceq \mathbf{R}'$ denote the claim that there exists a Tukey connection from $\mathbf{R}$ to $\mathbf{R}'$, and we let $\mathbf{R} \equiv \mathbf{R}'$ abbreviate $\mathbf{R} \preceq \mathbf{R}' \preceq \mathbf{R}$. Note that if $\langle \rho_-, \rho_+ \rangle$ witnesses $\mathbf{R} \preceq \mathbf{R}'$, then $\langle \rho_+, \rho_- \rangle$ witnesses $\mathbf{R}'^{\perp} \preceq \mathbf{R}^{\perp}$.

We will use Tukey connections to give an ordering between the norms of relational systems, through the following lemma.

Lemma 2.3.1 — *[Bla10, Theorem 4.9]*
If $\mathbf{R} \preceq \mathbf{R}'$, then $\|\mathbf{R}\| \leq \|\mathbf{R}'\|$ and $\|\mathbf{R}'^{\perp}\| \leq \|\mathbf{R}^{\perp}\|$.

We will on one occasion require a composition of several relational systems. Specifically we need a general form of the (categorical) product. We will state the definition and give two lemmas that generalise Theorem 4.11 of [Bla10] and give us the means to compute norms.

Let $\mathbf{R}_\alpha = \langle X_\alpha, Y_\alpha, R_\alpha \rangle$ be relational systems for each $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is a set of ordinals (but we will omit mention of $\mathcal{A}$ from now on for the sake of brevity). We define the categorical product as the system $\bigotimes_\alpha \mathbf{R}_\alpha = \langle \bigcup_\alpha (X_\alpha \times \{\alpha\}), \prod_\alpha Y_\alpha, Z \rangle$, where $(x, \alpha)\ Z\ \overline{y}$ iff $x\ R_\alpha\ \overline{y}(\alpha)$. Dually we have the categorical coproduct $\bigoplus_\alpha \mathbf{R}_\alpha = (\bigotimes_\alpha \mathbf{R}_\alpha^{\perp})^{\perp} = \langle \prod_\alpha X_\alpha, \bigcup_\alpha (Y_\alpha \times \{\alpha\}), N \rangle$, where $\overline{x}\ N\ (y, \alpha)$ iff $\overline{x}(\alpha)\ R_\alpha\ y$.

Lemma 2.3.2
$\|\bigotimes_\alpha \mathbf{R}_\alpha\| = \sup_\alpha \|\mathbf{R}_\alpha\|$. ◁

Proof. $(\leq)$ Let $Y'_\alpha \subseteq Y_\alpha$ be a witness for $|Y'_\alpha| = \|\mathbf{R}_\alpha\|$. Let $\lambda = \sup_\alpha \|\mathbf{R}_\alpha\|$ and let $\sigma_\alpha : \lambda \twoheadrightarrow Y'_\alpha$ be surjections. Define $\overline{y}_\xi : \alpha \mapsto \sigma_\alpha(\xi)$ and let $Y = \{\overline{y}_\xi \mid \xi \in \lambda\} \subseteq \prod_\alpha Y_\alpha$. If $x \in X_\alpha$, let $y \in Y'_\alpha$ such that $x\ R_\alpha\ y$ and find $\xi \in \lambda$ such that $y = \sigma_\alpha(\xi)$, then $(x, \alpha)\ Z\ \overline{y}_\xi$. Hence Y witnesses that $\|\bigotimes_\alpha \mathbf{R}_\alpha\| \leq \lambda$.

$(\geq)$ Let $Y \subseteq \prod_\alpha Y_\alpha$ be a witness for $|Y| = \|\bigotimes_\alpha \mathbf{R}_\alpha\|$, and let $Y'_\alpha = \{\overline{y}(\alpha) \mid \overline{y} \in Y\} \subseteq Y_\alpha$. Then for every $x \in X_\alpha$ there exists $\overline{y} \in Y$ such that $(x, \alpha)\ Z\ \overline{y}$, hence $\overline{y}(\alpha) \in Y'_\alpha$ has the property that $x\ R_\alpha\ \overline{y}(\alpha)$, thus $\|\mathbf{R}_\alpha\| \leq |Y'_\alpha| \leq |Y| = \|\bigotimes_\alpha \mathbf{R}_\alpha\|$. □

Lemma 2.3.3
$\|\bigoplus_\alpha \mathbf{R}_\alpha\| = \inf_\alpha \|\mathbf{R}_\alpha\|$. ◁

Proof. $(\leq)$ Let $Y \subseteq Y_\alpha$ be a witness for $|Y| = \|\mathbf{R}_\alpha\|$. Let $Y' = \{(y, \alpha) \mid y \in Y\}$. If $\overline{x} \in \prod_\alpha X_\alpha$, then there exists $y \in Y$ such that $\overline{x}(\alpha)\ R_\alpha\ y$ and thus $\overline{x}\ N\ (y, \alpha)$. Therefore $\|\bigoplus_\alpha \mathbf{R}_\alpha\| \leq |Y'| = |Y| = \|\mathbf{R}_\alpha\|$.

$(\geq)$ Let $Y \subseteq \bigcup_\alpha (Y_\alpha \times \{\alpha\})$ with $|Y| < \inf_\alpha \|\mathbf{R}_\alpha\|$. Let $Y'_\alpha = \{y \in Y_\alpha \mid (y, \alpha) \in Y\}$. Since $|Y'_\alpha| < \|\mathbf{R}_\alpha\|$ there is $x_\alpha \in X_\alpha$ such that $x_\alpha\ R_\alpha\ y$ for all $y \in Y'_\alpha$. Let $\overline{x} : \alpha \mapsto x_\alpha$, then $\overline{x}\ N\ (y, \alpha)$ for every $(y, \alpha) \in Y$. Thus $|Y| < \|\bigoplus_\alpha \mathbf{R}_\alpha\|$. □

2.4. The Cichoń Diagram

In this section we will first define the ten cardinal characteristics of the Cichoń diagram. After that, we will look at how (most of) these cardinal characteristics can be generalised to higher Baire spaces.

The Classical Cichoń Diagram

Each of the cardinal characteristics in the Cichoń diagram can be defined as the norm of a relational system, hence we will first consider some of the relevant relations. Eight of the cardinal characteristics of the Cichoń diagram are defined in terms of σ-ideals.

Definition 2.4.1

Let $\mathcal{I}$ be a σ-ideal on a space $\mathcal{X}$. Then we define the following two relational systems:

$$\mathsf{C}_{\mathcal{I}} = \langle \mathcal{X}, \mathcal{I}, \in \rangle \qquad \|\mathsf{C}_{\mathcal{I}}\| = \mathrm{cov}(\mathcal{I}) \qquad \left\|\mathsf{C}_{\mathcal{I}}^{\perp}\right\| = \mathrm{non}(\mathcal{I})$$

$$\mathsf{F}_{\mathcal{I}} = \langle \mathcal{I}, \mathcal{I}, \subseteq \rangle \qquad \|\mathsf{F}_{\mathcal{I}}\| = \mathrm{cof}(\mathcal{I}) \qquad \left\|\mathsf{F}_{\mathcal{I}}^{\perp}\right\| = \mathrm{add}(\mathcal{I}) \qquad \triangleleft$$

We can give a more intuitive description of these four cardinal characteristics and clarify their names as follows:

- The ***covering number*** $\mathrm{cov}(\mathcal{I})$ is the least cardinality of an $\mathcal{I}$-cover of $\mathcal{X}$, that is, a set $\mathcal{C} \subseteq \mathcal{I}$ with $\bigcup \mathcal{C} = \mathcal{X}$. To see that this is equivalent to the definition by norm, note that $\mathcal{C} \subseteq \mathcal{I}$ is an $\mathcal{I}$-cover iff for every $x \in \mathcal{X}$ there is $I \in \mathcal{C}$ with $x \in I$.
- The ***uniformity number***[8] $\mathrm{non}(\mathcal{I})$ is the least cardinality of an $\mathcal{I}$-positive set, that is, a set $X \subseteq \mathcal{X}$ with $X \notin \mathcal{I}$.
- The ***cofinality number*** $\mathrm{cof}(\mathcal{I})$ is the least cardinality of an ideal basis for $\mathcal{I}$, that is, of a set $\mathcal{B} \subseteq \mathcal{I}$ such that every $I \in \mathcal{I}$ has some $X \in \mathcal{B}$ with $I \subseteq X$. Equivalently, $\mathcal{B}$ is a $\subseteq$-cofinal subset of $\mathcal{I}$, explaining the name.
- The ***additivity number*** $\mathrm{add}(\mathcal{I})$ is the least cardinality of a set $\mathcal{J} \subseteq \mathcal{I}$ with $\bigcup \mathcal{J} \notin \mathcal{I}$, and intuitively answers the question how many $\mathcal{I}$-negligible sets should be added together to obtain an $\mathcal{I}$-positive set. To see that this is equivalent to the definition by norm, note that if $\mathcal{J} \subseteq \mathcal{I}$, then $\bigcup \mathcal{J} \notin \mathcal{I}$ iff for every $I \in \mathcal{I}$ we have $\bigcup \mathcal{J} \not\subseteq I$, or equivalently, iff there is some $J \in \mathcal{J}$ such that $J \not\subseteq I$.

Given a relational system $\mathsf{R} = \langle X, Y, R \rangle$ where the domain and range of R are reasonably clear, we will generally write $\mathfrak{d}(R) = \|\mathsf{R}\|$ and $\mathfrak{b}(R) = \left\|\mathsf{R}^{\perp}\right\|$. We consider this to be the case with the ***domination*** relation $\leq^*$ and the ***cofinal equality*** relation $=^\infty$, which we will only use on the space ${}^\omega\omega$ (and in higher context on ${}^\kappa\kappa$). Using the notation from Section 1.2, we have that $f \leq^* g$ for $f, g \in {}^\omega\omega$ if the set of $n \in \omega$ such that $f(n) > g(n)$ is bounded. On the other hand, $f =^\infty g$ for $f, g \in {}^\omega\omega$ holds if the set of $n \in \omega$ such that $f(n) = g(n)$ is cofinal (in ω). Our final two relations deal with the concept of ***localisation***, which requires us to define slaloms first.

If $h \in {}^\omega\omega$ is an unbounded increasing function that is nonzero everywhere, we define an h-***slalom*** to be a function φ with domain ω such that $|\varphi(n)| < h(n)$ for each $n \in \omega$.[9] If $f \in {}^\omega\omega$ and

[8] As for the confusing nomenclature, remember that an ultrafilter U on X is called *uniform* if $|X| = |X|$ for all $X \in U$, which is equivalent to saying that $\mathrm{non}(U^*) = |X|$ for U^* the (prime) ideal dual to U. Generalising this notion to non-maximal filters and cardinalities $< |X|$ gives us a uniformity number.

[9] Note that this definition differs from the usual definition: we define φ such that $|\varphi(\alpha)| < h(\alpha)$, instead of the traditional $|\varphi(\alpha)| \leq h(\alpha)$. Our definition is more versatile in the higher context. For example, if $h(\alpha)$ is a limit cardinal for each α, the resulting set of all slaloms φ with $|\varphi(\alpha)| < h(\alpha)$ cannot be expressed using the traditional definition. On the other hand, the set of all traditionally defined h-slaloms is the set of h^+-slaloms under our definition.

φ is an h-slalom, we say that f is ***localised*** by φ if the set of $n \in \omega$ such that $f(n) \notin \varphi(n)$ is bounded, that is, $f \in^* \varphi$. The set of all h-slaloms is denoted by Loc^h, and is another example of a perfect Polish space under the appropriate topology. We may also define ***antilocalisation***, where we say that f is ***antilocalised*** by φ if the set of $n \in \omega$ such that $f(n) \in \varphi(n)$ is bounded, that is, $f \in^\infty \varphi$.

Definition 2.4.2
We define the following relational systems:

$$\begin{aligned} \mathcal{D} &= \langle {}^\omega\omega, {}^\omega\omega, \leq^* \rangle & \|\mathcal{D}\| &= \mathfrak{d}(\leq^*) & \left\|\mathcal{D}^\perp\right\| &= \mathfrak{b}(\leq^*) \\ \mathcal{ED} &= \left\langle {}^\omega\omega, {}^\omega\omega, =^\infty \right\rangle & \|\mathcal{ED}\| &= \mathfrak{d}(=^\infty) & \left\|\mathcal{ED}^\perp\right\| &= \mathfrak{b}(=^\infty) \\ \mathcal{L}^h &= \left\langle {}^\omega\omega, \mathrm{Loc}^h, \in^* \right\rangle & \left\|\mathcal{L}^h\right\| &= \mathfrak{d}^h(\in^*) & \left\|\mathcal{L}^{h\perp}\right\| &= \mathfrak{b}^h(\in^*) \\ \mathcal{AL}^h &= \left\langle \mathrm{Loc}^h, {}^\omega\omega, \ni^\infty \right\rangle & \left\|\mathcal{AL}^h\right\| &= \mathfrak{d}^h(\ni^\infty) & \left\|\mathcal{AL}^{h\perp}\right\| &= \mathfrak{b}^h(\ni^\infty) \end{aligned} \qquad \triangleleft$$

We call $\mathfrak{d}(\leq^*)$ the ***dominating*** number, $\mathfrak{b}(\leq^*)$ the ***unbounding*** number, $\mathfrak{d}(=^\infty)$ the ***eventual di erence*** number and $\mathfrak{b}(=^\infty)$ the ***cofinal equality*** number[10]. Both $\mathfrak{d}^h(\in^*)$ and $\mathfrak{b}^h(\in^*)$ are sometimes called ***localisation*** numbers, but we believe it will be useful to distinguish between these classes of cardinals with separate terms. We therefore opt to call $\mathfrak{d}^h(\in^*)$ the ***h-localisation*** number, and $\mathfrak{b}^h(\in^*)$ the ***h-avoidance*** number, for the reason that a witness to $\mathfrak{b}^h(\in^*)$ is a set of functions $F \subseteq {}^\omega\omega$ that cannot be localised by a single h-slalom; for any specific h-slalom φ there is at least one member f of F that ***avoids*** φ by having $f(n) \notin \varphi(n)$ for cofinally many $n \in \omega$. Finally this leaves the cardinals $\mathfrak{d}^h(\ni^\infty)$, which we will call the ***h-antiavoidance*** number and $\mathfrak{b}^h(\ni^\infty)$, which we will call the ***h-antilocalisation*** number.

Note that we opted to use the relations $=^\infty$ and $\ni^\infty$ instead of $=^\infty$ and $\in^\infty$ in the above relational systems. Due to duality of relational systems, this is only relevant for the names of our cardinals (in deciding which of the two norms is the $\mathfrak{d}$-cardinal, and which the $\mathfrak{b}$-cardinal). The reason for defining the cardinal characteristics with the negated relations, is that we will later see that $\mathfrak{d}(=^\infty)$ and $\mathfrak{d}^h(\ni^\infty)$ bear more similarities to $\mathfrak{d}(\leq^*)$ and $\mathfrak{d}^h(\in^*)$ than to $\mathfrak{b}(\leq^*)$ and $\mathfrak{b}^h(\in^*)$.

We should mention that our notation is not entirely standard. Usually $\mathfrak{b}(\leq^*)$ and $\mathfrak{d}(\leq^*)$ are simply written as $\mathfrak{b}$ and $\mathfrak{d}$. Although classically the other cardinals defined using relations do not give us new cardinal characteristics (see Fact 2.4.3), we will consider many variants of them that do produce new cardinal characteristics. We therefore opted to write the relation in each of our cardinal characteristics for easy comparison.

We also note that (anti)localisation and (anti)avoidance cardinals have been described using a variety of names and notations, especially in the bounded context that we will introduce in the next chapter, where not only a parameter h for the size of the sets in the slalom, but also a parameter b is used to describe the bounded Baire space $\prod b$. In previous literature, the

[10] In the classical setting, the relation of being cofinally equal is often referred to as being *infinitely equal*, but this will not be precise enough in the higher context.

notations $\mathsf{c}^{\forall}_{b,h}$ and $\mathsf{v}^{\forall}_{b,h}$ have also frequently been used for $\mathsf{d}^{b,h}(\in^*)$ and $\mathsf{b}^{b,h}(\in^*)$ respectively, while $\mathsf{c}^{\exists}_{b,h}$ and $\mathsf{v}^{\exists}_{b,h}$ have been used for $\mathsf{b}^{b,h}(\ni^\infty)$ and $\mathsf{d}^{b,h}(\ni^\infty)$ respectively. For example, this notation was used in [Kel08, KO14, KM22, CKM21]. Here c stands for a **c**over of $\prod b$ with slaloms by the relations $\in^*$ or $\in^\infty$, and v stands for a**v**oidance by or e**v**asion by a single slalom. We believe that ***avoidance*** forms the better antonym to ***localisation***, as it prevents confusion with other cardinal characteristics known as evasion cardinals (such as described in [Bla10, Section 10]). Our choice to use the notation $\mathsf{d}^{b,h}(\in^*)$ over $\mathsf{c}^{\forall}_{b,h}$ has the main benefit that the relevant relational system can be deduced from our notation.

The Cichoń diagram consists of the ten cardinal characteristics that are drawn below. Next to the bounds $\aleph_1$ and $\mathfrak{c} = 2^{\aleph_0}$, the four outermost cardinals are defined in terms of the Lebesgue null ideal $\mathcal{N}$, the two innermost cardinals are the dominating and unbounding numbers, and the remaining four cardinals are defined in terms of the meagre ideal $\mathcal{M}$. The diagram shows which relations between these cardinal characteristics are provable, where an arrow $\mathsf{x} \to \mathsf{y}$ implies that $\mathsf{x} \leq \mathsf{y}$ is provable in ZFC.

$$
\begin{array}{ccccccccccc}
 & & \operatorname{cov}(\mathcal{N}) & \longrightarrow & \operatorname{non}(\mathcal{M}) & \longrightarrow & \operatorname{cof}(\mathcal{M}) & \longrightarrow & \operatorname{cof}(\mathcal{N}) & \longrightarrow & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathsf{b}(\leq^*) & \longrightarrow & \mathsf{d}(\leq^*) & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\aleph_1 & \longrightarrow & \operatorname{add}(\mathcal{N}) & \longrightarrow & \operatorname{add}(\mathcal{M}) & \longrightarrow & \operatorname{cov}(\mathcal{M}) & \longrightarrow & \operatorname{non}(\mathcal{N}) & &
\end{array}
$$

We will not give proofs of the relations drawn in the Cichoń diagram, and refer to [BJ95] as a detailed reference for the Cichoń diagram. Our focus lies on the higher context, but we will complete this section by mentioning some additional results regarding the classical Cichoń diagram, to have a basis to compare to.

The following equivalences are due to Miller and Bartoszyński, and give us combinatorial definitions for some of the cardinals that have been expressed in terms of an ideal.

Fact 2.4.3 — *[Bar87]* [11]

We have the following combinatorial descriptions (for any cofinally increasing $h \in {}^\omega\omega$):

$$
\begin{aligned}
\operatorname{non}(\mathcal{M}) &= \mathsf{b}(=^\infty) = \mathsf{b}^h(\ni^\infty),\\
\operatorname{cov}(\mathcal{M}) &= \mathsf{d}(=^\infty) = \mathsf{d}^h(\ni^\infty),\\
\operatorname{add}(\mathcal{N}) &= \mathsf{b}^h(\in^*),\\
\operatorname{cof}(\mathcal{N}) &= \mathsf{d}^h(\in^*).
\end{aligned}
$$

It follows from this theorem that the choice of parameter $h \in {}^\omega\omega$ is hardly relevant for any of the cardinal characteristics $\mathsf{b}^h(\in^*)$, $\mathsf{d}^h(\in^*)$, $\mathsf{b}^h(\ni^\infty)$ or $\mathsf{d}^h(\ni^\infty)$: any cofinally increasing h will

[11] Miller showed in [Mil81, Theorems 1.3 & 1.4] relations between $\operatorname{cov}(\mathcal{M})$, $\operatorname{non}(\mathcal{M})$ and $=^\infty$, and Bartoszyński introduced (anti)localisation and proved the equalities mentioned in the lemma, albeit with very different notation. See also [BJ95, Theorems 2.3.9, 2.4.1 & 2.4.7] for proofs with modern notation.

result in the same cardinal characteristic. This contrasts with the higher Baire space ${}^{\kappa}\kappa$ with κ inaccessible, where many cardinals of the forms $\mathfrak{d}^h_\kappa(\in^*)$ can be different from each other (which is the subject of Chapter 5).

We should also mention that two cardinal characteristics of the Cichoń diagram are dependent on the others by the following lemma, originally due to Truss, Miller and Fremlin:

Fact 2.4.4 — *[Tru77, Mil81, Fre84]* [12]

We have:

$$\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}(\leq^*), \mathrm{cov}(\mathcal{M})\} \text{ and}$$
$$\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}(\leq^*), \mathrm{non}(\mathcal{M})\}.$$

As for the other eight cardinal characteristics of the Cichoń diagram, they appear to be as independent as is possible in the following sense: any assignment of the cardinalities $\aleph_1$ and $\aleph_2$ to the cardinals of the Cichoń Diagram that does not contradict the relations given by the arrows or Fact 2.4.4, is consistent.[13] This implies that the Cichoń diagram is complete, in the sense that no other arrows are missing from the diagram.

Furthermore, a problem known as ***Cichoń's Maximum*** asks whether it is consistent that all eight (independent) cardinal characteristics of the Cichoń diagram are mutually different from each other. Cichoń's Maximum has been shown to be consistent, first under the assumption of four strongly compact cardinals by Goldstern, Kellner & Shelah [GKS19], and later without any large cardinal assumptions by Goldstern, Kellner, Mejía & Shelah [GKMS22]. Other recent efforts have led to the consistency of Cichoń's Maximum with several additional cardinal characteristics that are not included in the Cichoń diagram, and towards reaching Cichoń's Maximum with different orderings of the eight cardinal characteristics than were given in the above-mentioned papers.[14]

The Higher Cichoń Diagram

Let us assume for this section (as we do in general) that κ is an uncountable cardinal, then ${}^{\kappa}\kappa$ is a higher Baire space. We have previously witnessed that we can define the $\leq\kappa$-complete ideal $\mathcal{M}_\kappa$ of κ-meagre subsets of ${}^{\kappa}\kappa$. We may suitably generalise relations such as $\leq^*$ as well, where $f \leq^* g$ with $f, g \in {}^{\omega}\omega$ and the definition $\forall^\infty n \in \omega(f(n) \leq g(n))$ is generalised to $f \leq^* g$ with $f, g \in {}^{\kappa}\kappa$ and the definition $\forall^\infty \alpha \in \kappa(f(\alpha) \leq g(\alpha))$. Finally, we may generalise slaloms as well, where an h-slalom for some cofinally increasing $h \in {}^{\kappa}\kappa$ is a function φ with domain κ such that $|\varphi(\alpha)| < h(\alpha)$ for each $\alpha \in \kappa$. The set of all such h-slaloms will be denoted by Loc^h_κ, to emphasise that we are working in the higher context.

We may generalise the cardinal characteristics defined so far to ${}^{\kappa}\kappa$, as per the definition below.

[12] Truss [Tru77, Theorem 6.5] constructed an embedding between Boolean algebras, from which $\mathrm{add}(M)$ $\min f\mathrm{b}(\;{}^*), \mathrm{cov}(M)g$ follows, Miller [Mil81, Theorem 1.2] proved the converse, and Fremlin remarked that one could also show the dual result about $\mathrm{cof}(M)$. See also [BJ95, Corollary 2.2.9 & Theorem 2.2.11] for proofs with modern notation.

[13] See [BJ95, Sections 7.5 and 7.6] for each of these cases.

[14] In fact, for many specific orderings Cichoń's Maximum is still an open problem.

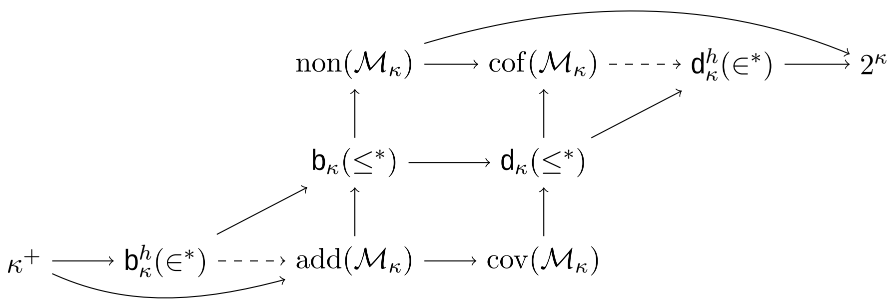
Definition 2.4.5

We define the following relational systems:

$$
\begin{aligned}
\mathsf{C}_{\mathcal{M}_\kappa} &= \langle {}^\kappa\kappa, \mathcal{M}_\kappa, \in\rangle & \|\mathsf{C}_{\mathcal{M}_\kappa}\| &= \operatorname{cov}(\mathcal{M}_\kappa) & \left\|\mathsf{C}_{\mathcal{M}_\kappa}^\perp\right\| &= \operatorname{non}(\mathcal{M}_\kappa)\\
\mathsf{F}_{\mathcal{M}_\kappa} &= \langle \mathcal{M}_\kappa, \mathcal{M}_\kappa, \subseteq\rangle & \|\mathsf{F}_{\mathcal{M}_\kappa}\| &= \operatorname{cof}(\mathcal{M}_\kappa) & \left\|\mathsf{F}_{\mathcal{M}_\kappa}^\perp\right\| &= \operatorname{add}(\mathcal{M}_\kappa)\\
\mathsf{D}_\kappa &= \langle {}^\kappa\kappa, {}^\kappa\kappa, \leq^*\rangle & \|\mathsf{D}_\kappa\| &= \mathsf{d}_\kappa(\leq^*) & \left\|\mathsf{D}_\kappa^\perp\right\| &= \mathsf{b}_\kappa(\leq^*)\\
\mathsf{ED}_\kappa &= \langle {}^\kappa\kappa, {}^\kappa\kappa, =^\infty\rangle & \|\mathsf{ED}_\kappa\| &= \mathsf{d}_\kappa(=^\infty) & \left\|\mathsf{ED}_\kappa^\perp\right\| &= \mathsf{b}_\kappa(=^\infty)\\
\mathsf{L}_\kappa^h &= \left\langle {}^\kappa\kappa, \mathrm{Loc}_\kappa^h, \in^*\right\rangle & \left\|\mathsf{L}_\kappa^h\right\| &= \mathsf{d}_\kappa^h(\in^*) & \left\|\mathsf{L}_\kappa^{h\perp}\right\| &= \mathsf{b}_\kappa^h(\in^*)\\
\mathsf{AL}_\kappa^h &= \left\langle \mathrm{Loc}_\kappa^h, {}^\kappa\kappa, \ni^\infty\right\rangle & \left\|\mathsf{AL}_\kappa^h\right\| &= \mathsf{d}_\kappa^h(\ni^\infty) & \left\|\mathsf{AL}_\kappa^{h\perp}\right\| &= \mathsf{b}_\kappa^h(\ni^\infty)
\end{aligned}
\qquad \triangleleft
$$

We name these cardinals as follows: for the cardinals defined in terms of the κ-meagre ideal, we call $\operatorname{cov}(\mathcal{M}_\kappa)$ the ***covering number of the κ-meagre ideal*** (and similar for the other three). We will name $\mathsf{d}_\kappa(\leq^*)$, $\mathsf{b}_\kappa(\leq^*)$, $\mathsf{d}_\kappa(=^\infty)$ and $\mathsf{b}_\kappa(=^\infty)$ similar to their ${}^\omega\omega$ counterpart, but with a prefix κ. For example, $\mathsf{d}_\kappa(\leq^*)$ is called the ***κ-dominating number***. Finally, since the fact that we are working in the higher Baire space ${}^\kappa\kappa$ is already clear from the domain of $h \in {}^\kappa\kappa$, and since we want to avoid too many prefixes, we will simply keep referring to $\mathsf{d}_\kappa^h(\in^*)$ as the ***h-localisation cardinal***, and similar $\mathsf{b}_\kappa^h(\in^*)$, $\mathsf{d}_\kappa^h(\ni^\infty)$ and $\mathsf{b}_\kappa^h(\ni^\infty)$.

As mentioned in Section 2.2, there is no clear way to generalise the Lebesgue null ideal. However, in the case of $\operatorname{add}(\mathcal{N})$ and $\operatorname{cof}(\mathcal{N})$, we can replace these cardinal characteristics in the higher context by their combinatorially defined counterparts, the h-localisation and h-avoidance numbers from Fact 2.4.3. This yields the higher Cichoń diagram given below, where the dashed arrows require κ to be (strongly) inaccessible[15]:



2.5. Relations

We will briefly go through the ZFC-results known about the cardinals in the higher Cichoń diagram for the sake of completeness. We will not go into detail, and only give a cursory sketch of the way these results could be proved. We will then conclude this section with a succinct

[15] We will frequently assume κ is strongly inaccessible, and will generally simply say "inaccessible", omitting "strongly".

overview of some independence results to show that most of the cardinals of the higher Cichoń diagram are distinct from each other. Later, in Chapter 4, we will look at these independence results in more detail. For a thorough overview of the higher Cichoń diagram, a good starting point is [BBTFM18].

Our focus will largely be on the case where we assume that κ is inaccessible. We use this section to explain *why* we make this assumption, instead of letting κ be any regular uncountable cardinal.

ZFC-results

The centre six cardinal characteristics of the higher Cichoń diagram behave very similarly to their classical counterparts, and the proofs of the given relations are very much analogous.

For instance, an (almost trivial) Tukey connection $\mathsf{F}_I^\perp \preceq \mathsf{C}_I$, which works for any ideal I and thus does not depend on any properties specific to $\mathcal{M}_\kappa$, gives us:

Fact 2.5.1 — *Folklore*
$\mathrm{add}(\mathcal{M}_\kappa) \leq \mathrm{cov}(\mathcal{M}_\kappa)$ and $\mathrm{non}(\mathcal{M}_\kappa) \leq \mathrm{cof}(\mathcal{M}_\kappa)$.

A Tukey connection $\mathsf{D}_\kappa^\perp \preceq \mathsf{D}_\kappa$ follows easily from the observation that $g \leq^* f$ implies $f+1 \leq^* g$ and provides:

Fact 2.5.2 — *Folklore*
$\mathfrak{b}_\kappa(\leq^*) \leq \mathfrak{d}_\kappa(\leq^*)$.

The cardinals $\mathfrak{b}_\kappa(\leq^*)$ and $\mathfrak{d}_\kappa(\leq^*)$ are related to $\mathrm{non}(\mathcal{M}_\kappa)$ and $\mathrm{cov}(\mathcal{M}_\kappa)$ via the intermediate cardinal characteristics $\mathfrak{b}_\kappa(=^\infty)$ and $\mathfrak{d}_\kappa(=^\infty)$, provable through a sequence of Tukey connections $\mathsf{C}_{\mathcal{M}_\kappa} \preceq \mathsf{ED}_\kappa \preceq \mathsf{D}_\kappa$, using a similar method as in Fact 2.4.3:

Fact 2.5.3 — *Folklore, based on [Mil81, Bar87] for ${}^\omega\omega$*
$\mathrm{cov}(\mathcal{M}_\kappa) \leq \mathfrak{d}_\kappa(=^\infty) \leq \mathfrak{d}_\kappa(\leq^*)$ and $\mathfrak{b}_\kappa(\leq^*) \leq \mathfrak{b}_\kappa(=^\infty) \leq \mathrm{non}(\mathcal{M}_\kappa)$.

Remember that classically we have $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}(=^\infty)$ and $\mathfrak{b}(=^\infty) = \mathrm{non}(\mathcal{M})$, by Fact 2.4.3. In the higher case, we can prove equality of the norms only for inaccessible κ, which was done by Landver for $\mathrm{cov}(\mathcal{M}_\kappa)$ and by Blass, Hyttinen and Zhang for $\mathrm{non}(\mathcal{M}_\kappa)$:

Fact 2.5.4 — *[Lan92, Section 1],[BHZ07, Section 4]*
If κ is inaccessible, $\mathrm{cov}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa(=^\infty)$ and $\mathrm{non}(\mathcal{M}_\kappa) = \mathfrak{b}_\kappa(=^\infty)$.

In the accessible case[16] the situation is quite different, since we have the following results:

Fact 2.5.5 — *[Hyt06, Definition 13 and following text], [MS04, Theorem 4.6]*
If κ is successor, then $\mathfrak{b}_\kappa(=^\infty) = \mathfrak{b}_\kappa(\leq^*)$, and if furthermore $2^{<\kappa} = \kappa$, then $\mathfrak{d}_\kappa(=^\infty) = \mathfrak{d}_\kappa(\leq^*)$.

Fact 2.5.6 — *[Lan92, Lemma 1.3]*
If $2^{<\kappa} > \kappa$ for regular κ, then $\mathrm{add}(\mathcal{M}_\kappa) = \mathrm{cov}(\mathcal{M}_\kappa) = \kappa^+$.

[16] We will call a cardinal *accessible* if it is uncountable but not inaccessible.

Fact 2.5.7 — *[BHZ07, Proposition 4.15]*
$2^{<\kappa} \leq \mathrm{non}(\mathcal{M}_\kappa)$.

Fact 2.5.8 — *[Bre17, Proposition 2]*
$2^{<\kappa} < \mathrm{cof}(\mathcal{M}_\kappa)$.

These results allow for the consistency of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa(=^\infty)$ and $\mathfrak{b}_\kappa(=^\infty) < \mathrm{non}(\mathcal{M}_\kappa)$ as well. In some accessible cases the value of some cardinal characteristics have absolute values (under cardinal preserving forcing notions) and others can be influenced by the value of 2^λ for some $\lambda < \kappa$. This is one reason why we will generally assume that κ is inaccessible.

The classical results from Fact 2.5.9 generalise to the higher case as follows:

Fact 2.5.9 — *[Bre22, Corollary 4]*
For κ regular uncountable we have:

$$\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}_\kappa(\leq^*), \mathrm{cov}(\mathcal{M}_\kappa\} \text{ and}$$
$$\mathrm{cof}(\mathcal{M}_\kappa) \geq \max\{\mathfrak{d}_\kappa(\leq^*), \mathrm{non}(\mathcal{M}_\kappa)\}.$$

In the latter, equality holds if $2^{<\kappa} = \kappa$.

In particular, the above characterisations of $\mathrm{add}(\mathcal{M}_\kappa)$ and $\mathrm{cof}(\mathcal{M}_\kappa)$ hold for inaccessible κ.

For inaccessible κ, if we consider $\mathfrak{b}^h_\kappa(\in^*)$ and $\mathfrak{d}^h_\kappa(\in^*)$, there exist Tukey connections $\boldsymbol{ED}_\kappa \preceq \boldsymbol{L}^h_\kappa$ and $\boldsymbol{D}_\kappa \preceq \boldsymbol{L}^h_\kappa$. Together with Fact 2.5.9, this implies:

Fact 2.5.10 — *[BBTFM18, Corollary 41 & Observation 36]*
If κ is inaccessible, $\mathfrak{b}^h_\kappa(\in^*) \leq \mathrm{add}(\mathcal{M}_\kappa)$ and $\mathrm{cof}(\mathcal{M}_\kappa) \leq \mathfrak{d}^h_\kappa(\in^*)$.

As mentioned above, these two inequalities require that the norms of $\boldsymbol{ED}_\kappa$ and $\boldsymbol{C}_{\mathcal{M}_\kappa}$ are equal. Consequently, for accessible cardinals, we can only prove that $\mathfrak{b}^h_\kappa(\in^*) \leq \mathfrak{b}_\kappa(\leq^*)$ and $\mathfrak{d}_\kappa(\leq^*) \leq \mathfrak{d}^h_\kappa(\in^*)$, giving another reason why we prefer κ to be inaccessible.

Independence Results

This subsection serves as a very brief overview of independence proofs concerning the previously defined cardinal characteristics, without giving details to how these results were proved. We will mention the forcing notions and techniques used without definition. Later, in Chapter 4 we will discuss most of these forcing constructions in detail. This subsection therefore mostly serves as a summary of results on the higher Cichoń diagram for those readers already familiar with independence results of the classical Cichoń diagram and can safely be ignored by all other readers.

Splitting the higher Cichoń diagram with a "vertical" separation (see Figure 2.1 below) is relatively easy, and can be achieved with higher variants of Cohen, Hechler and localisation forcing.

The κ-Cohen model

Adding λ-many κ-Cohen reals (i.e. generics for κ-Cohen forcing) via a $<\kappa$-support iteration, where $\lambda > \kappa^+$, over a model $\mathbf{V}$ "$2^\kappa = \kappa^+$" will result in a model where $\kappa^+ = \mathrm{non}(\mathcal{M}_\kappa) < \lambda \leq \mathrm{cov}(\mathcal{M}_\kappa)$. If $\lambda = \kappa^{++}$, we will call the resulting model the *κ-Cohen model*. This result can probably be attributed to folklore, as the argument is exactly as its ${}^\omega\omega$-counterpart.

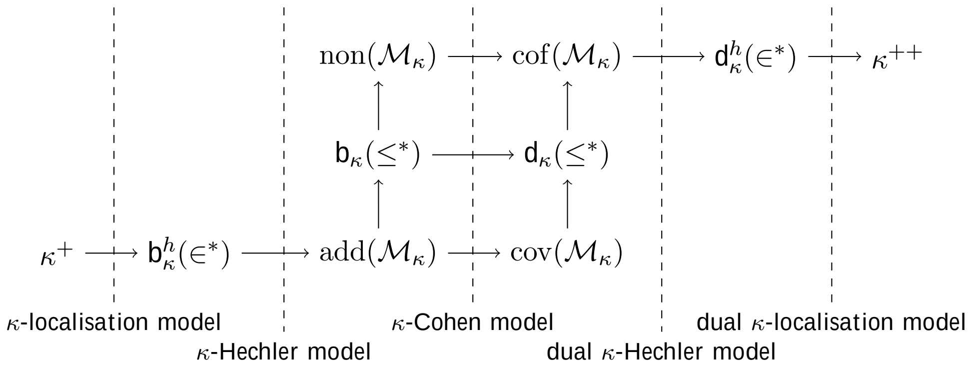


Figure 2.1: The higher Cichoń diagram in several forcing extensions for inaccessible κ.

Models from κ-Hechler forcing

The consistency of $\lambda = \mathsf{b}_\kappa(\leq^*) < \mathsf{d}_\kappa(\leq^*) = \mu$ for $\kappa^+ \leq \lambda$ is subject of the first paper on cardinal characteristics on higher Baire spaces, by Cummings & Shelah [CS95], where it is shown that the only requirements are that κ and λ are regular and $\lambda \leq \mathrm{cf}(\mu)$. The forcing notion used is a special iteration (along a second poset) of κ-Hechler forcing, which is a forcing notion that adds dominating κ-reals.

Like Hechler forcing satisfying σ-centredness, κ-Hechler forcing satisfies a property that we will call $(\kappa, <\kappa)$-centredness (see Definition 4.1.9). Brendle, Brooke-Taylor, Friedman & Montoya [BBTFM18, Lemma 55] showed that the property of being $(\kappa, <\kappa)$-centred can be preserved under $<\kappa$-support iteration of length $<(2^\kappa)^+$ if the forcing notions are additionally "closed with canonical bounds"[17]. $(\kappa, <\kappa)$-centred forcing notions do not affect $\mathsf{b}^h_\kappa(\in^*)$ and $\mathsf{d}^h_\kappa(\in^*)$.

It happens to be the case that κ-Hechler forcing indeed is $(\kappa, <\kappa)$-centred with canonical bounds, allowing us to produce a model of $\mathsf{b}^h_\kappa(\in^*) = \kappa^+ < \lambda = \mathrm{add}(\mathcal{M}_\kappa)$ by starting with a model $\mathbf{V}$ "$2^\kappa = \kappa^+$" and doing a $<\kappa$-support iteration of κ-Hechler forcing of length λ. If $\lambda = \kappa^{++}$ we will call this the *κ-Hechler model*.

Dually, if we start with a model $\mathbf{V}$ "$\mathsf{d}_\kappa(\in^*) = \lambda = 2^\kappa$" and we do a $<\kappa$-support iteration of κ-Hechler forcing of length κ^+, we end up with $\mathrm{cof}(\mathcal{M}_\kappa) = \kappa^+ < \lambda = \mathsf{d}^h_\kappa(\in^*)$. If $\lambda = \kappa^{++}$, we call the resulting model the *dual κ-Hechler model*.

[17] This is called "finely closed" in [BGS20, Section 2.3]

Models from κ-Localisation forcing

Let $\mathsf{id} : \alpha \mapsto |\alpha|$. The consistency of $\kappa^+ < \lambda = \mathsf{b}^{\mathsf{id}}_\kappa(\in^*)$ or of $\kappa^+ = \mathsf{d}^{\mathsf{id}}_\kappa(\in^*) < \lambda = 2^\kappa$ can be shown using κ-localisation forcing, as is done in [BBTFM18, Proposition 52]. In the same way as with κ-Hechler forcing, if we force with a $<\kappa$-support iteration of κ-localisation forcing of length λ, we get $\kappa^+ < \lambda = \mathsf{b}^{\mathsf{id}}_\kappa(\in^*)$ in the resulting model. If $\lambda = \kappa^{++}$, we call this the *κ-localisation model*.

If we instead start with $\mathbf{V}$ “$\lambda = 2^\kappa$” and do a $<\kappa$-support iteration of κ-localisation forcing of length κ^+, we obtain a model for $\kappa^+ = \mathsf{d}^{\mathsf{id}}_\kappa(\in^*) < \lambda = 2^\kappa$. If $\lambda = \kappa^{++}$, this is called the *dual κ-localisation model*.

The same kinds of models can be used to show this result for $\mathsf{b}^h_\kappa(\in^*)$ and $\mathsf{d}^h_\kappa(\in^*)$ with $h \in {}^\kappa\kappa$ any cofinally increasing κ-real.

The κ-Sacks model

Whereas in the context of ${}^\omega\omega$ the parameter h has no influence over the localisation and avoidance cardinals, it is shown in [BBTFM18, Theorem 70] that $\mathsf{d}^{\mathsf{pow}}_\kappa(\in^*) < \mathsf{d}^{\mathsf{id}}_\kappa(\in^*)$ is consistent, where[18] $\mathsf{id} : \alpha \mapsto |\alpha|^+$ and $\mathsf{pow} : \alpha \mapsto (2^{|\alpha|})^+$, since it holds in the κ-Sacks model. Similar forcing notions can be used to separate more cardinals of the form $\mathsf{d}^h_\kappa(\in^*)$, as we will discuss in Chapter 5.

Bounded κ-Hechler forcing

Finally, we will mention that not only “vertical” consistency results are known. Shelah [She20] has given a construction of a model using a bounded version of κ-Hechler forcing to prove the consistency of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathsf{d}_\kappa(\leq^*)$. In order for the forcing notion to have the right properties, one needs to assume that κ is weakly compact, and in order to preserve weakly compact cardinals under iteration, a type of Laver indestructible supercompact cardinals are needed.

The accessible case

If we let go of the requirement that $2^{<\kappa} = \kappa$, the κ-meagre ideal behaves less nicely, as witnessed by previously mentioned ZFC-results. There are several more consistency results known in this context.

In [BHZ07] it is shown that $\mathsf{b}_\kappa(=^\infty) < \mathrm{non}(\mathcal{M}_\kappa)$ is consistent with $2^{<\kappa} > \kappa$. The consistency of $\mathsf{d}_\kappa(\leq^*) < \mathrm{non}(\mathcal{M}_\kappa)$ and of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathsf{b}_\kappa(\leq^*)$ with $2^{<\kappa} > \kappa$ is also relatively simple, and is described in [BBTFM18, page 12], whereas [BBTFM18, Theorem 49, Proposition 53] prove the consistency of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathsf{d}_\kappa(=^\infty)$ with $2^{<\kappa} > \kappa$, as well as the consistency of the following statements:

$$\kappa^+ = \mathrm{cov}(\mathcal{M}_\kappa) < \mathsf{b}_\kappa(=^\infty) = \mathsf{d}_\kappa(=^\infty) < \mathrm{non}(\mathcal{M}_\kappa) = 2^{<\kappa} = 2^\kappa < \mathrm{cof}(\mathcal{M}_\kappa),$$
$$\kappa^+ = \mathrm{cov}(\mathcal{M}_\kappa) < \mathsf{b}^h_\kappa(\in^*) = \mathsf{d}^h_\kappa(\in^*) < \mathrm{non}(\mathcal{M}_\kappa) = 2^{<\kappa} = 2^\kappa < \mathrm{cof}(\mathcal{M}_\kappa).$$

Finally, Brendle [Bre22, Theorem 7] has shown that $\kappa < 2^{<\kappa} \leq \mathrm{non}(\mathcal{M}_\kappa) \leq 2^\kappa$ is consistent.

[18] Taking the successor cardinal $j\alpha j^+$ is necessary since we define our h-slaloms with a strict bound.

2.6. Open Questions

The main open question regarding the higher Cichoń diagram, is how we can separate in a "horizontal" manner. Not much is known apart from Shelah's proof of the consistency of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa(\leq^*)$, leading to the following series of closely related questions.

Question 2.6.1
Are any of the following consistent with κ inaccessible?

1. $\mathrm{cov}(\mathcal{M}_\kappa) < \mathrm{non}(\mathcal{M}_\kappa)$
2. $\mathfrak{b}_\kappa(\leq^*) < \mathrm{non}(\mathcal{M}_\kappa)$
3. $\mathfrak{d}_\kappa(\leq^*) < \mathrm{cof}(\mathcal{M}_\kappa)$ (or equivalently $\mathfrak{d}_\kappa(\leq^*) < \mathrm{non}(\mathcal{M}_\kappa)$)
4. $\mathrm{add}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa(\leq^*)$ (or equivalently $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{b}_\kappa(\leq^*)$) ◁

As far as the consistency of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa(\leq^*)$ is concerned, the proof uses a form of Laver indestructibility to preserve the property that κ is weakly compact. In order to get rid of the supercompactness assumption, one would need to find a proof that does not require the preservation of a weakly compact cardinal, and thus a different forcing in the single step.

Question 2.6.2
Is $\mathrm{cov}(\mathcal{M}_\kappa) < \mathfrak{d}_\kappa(\leq^*)$ provable without the use of a supercompact cardinal? ◁

For inaccessible κ, we saw that $\mathfrak{d}_\kappa(=^\infty) = \mathrm{cov}(\mathcal{M}_\kappa)$ and $\mathfrak{b}_\kappa(=^\infty) = \mathrm{non}(\mathcal{M}_\kappa)$, whereas for successor κ we saw that $\mathfrak{b}_\kappa(=^\infty) = \mathfrak{b}_\kappa(\leq^*)$ is provable. For $\mathfrak{d}_\kappa(=^\infty) = \mathfrak{d}_\kappa(\leq^*)$, one needed the additional assumption that $2^{<\kappa} = \kappa$, thus it is natural to ask whether this additional assumption is necessary.

Question 2.6.3 — *[MS04]*
If κ is successor and $2^{<\kappa} > \kappa$, is $\mathfrak{d}_\kappa(=^\infty) < \mathfrak{d}_\kappa(\leq^*)$ consistent? ◁

Finally, we mention a question related to $\mathrm{cof}(\mathcal{M}_\kappa)$. In [Bre22, Corollary 4], a connection between $\mathrm{cof}(\mathcal{M}_\kappa)$, $\mathrm{non}(\mathcal{M}_\kappa)$ and bounded variants of the dominating number is proved, making the following question potentially relevant to the subject matter of the next chapter.

Question 2.6.4 — *[Bre22, Question 11]*
Is $\kappa < 2^{<\kappa} < \mathrm{cof}(\mathcal{M}_\kappa) < 2^{2^{<\kappa}}$ consistent? ◁

3

Bounded Cardinal Characteristics on Higher Baire Spaces

The focus of this chapter will be variants of the cardinal characteristics that we have defined in the previous chapter in the context of bounded subspaces of higher Baire spaces. Some results in this chapter can be basically described as the result of substituting ω by κ in some known results on ${}^{\omega}\omega$. However, for many other results, the difference is significant due to additional structure on ${}^{\kappa}\kappa$ (in particular the structure of stationary sets). We will also introduce bounded forms of the dominating and unbounded numbers that have no analogue on ${}^{\omega}\omega$. Finally we will decide which parameters (in the form of bounds for the space, and for the size of slaloms) will result in trivial cardinals, and state several unexpected and interesting questions resulting from this inquiry.

Nota Bene! In this chapter we will assume that κ is a (strongly) inaccessible cardinal without further mention. This assumption will **not** be stated in the theorems and lemmas, but is often required. Additionally, we will assume that the functions in ${}^{\kappa}\kappa$ denoted by the letters b, h are cofinal increasing nonfinite cardinal functions, that is, b is increasing, $\mathrm{ran}(b)$ is cofinal in κ, and $b(\alpha)$ is an infinite cardinal for each $\alpha \in \kappa$ (and similar for h). This also extends to indexed or accented functions using the symbols b, h, such as b_ξ, h', and so on.

3.1. Bounded Higher Baire Spaces

Given a function $b \in {}^{\kappa}\kappa$, we consider the product space $\prod b = \prod_{\alpha\in\kappa} b(\alpha)$ where each $b(\alpha)$ has the discrete topology and $\prod b$ once again has the $<\kappa$-box topology, hence the topology on $\prod b$ is generated by sets $[s] = \{f \in \prod b \mid s \subseteq f\}$ with $s \in \prod_{<\kappa} b$, where $\prod_{<\kappa} b$ denotes the set of initial segments of functions in $\prod b$, as per Section 1.2. One can easily see that $\prod b$ is a closed subspace of ${}^{\kappa}\kappa$. We will refer to spaces of the form $\prod b$ as ***bounded higher Baire spaces***.

Remember that a set $X \subseteq \prod b$ is ***nowhere dense*** if for every basic open $[s] \subseteq \prod b$ there exists a basic open $[t] \subseteq [s]$ such that $[t] \cap X =$?. It is easy to show that the closure of a nowhere dense set is nowhere dense. The complement of a closed nowhere dense set is an ***open dense set***. We will write $\mathcal{N}^b_\kappa$ for the family of nowhere dense subsets of $\prod b$.

Also remember that a set X is ***κ-meagre*** if there exists a family $\{N_\alpha \mid \alpha \in \kappa\}$ of nowhere dense sets such that $X = \bigcup_{\alpha\in\kappa} N_\alpha$. Let $\mathcal{M}^b_\kappa$ be the set of κ-meagre subsets of $\prod b$.

It is easy to show that families $\mathcal{M}^b_\kappa$ (like $\mathcal{M}_\kappa$) are $\leq\kappa$-complete proper ideals. Moreover, by the following lemma, the choice of $b \in {}^{\kappa}\kappa$ does not matter for the value of the cardinal characteristics $\mathrm{add}(\mathcal{M}^b_\kappa)$, $\mathrm{cof}(\mathcal{M}^b_\kappa)$, $\mathrm{cov}(\mathcal{M}^b_\kappa)$ and $\mathrm{non}(\mathcal{M}^b_\kappa)$, and indeed, we could work with these cardinal characteristics as evaluated over $\mathcal{M}_\kappa$ without loss of generality.

Lemma 3.1.1 — *Folklore*[1]

There exists $M \in \mathcal{M}^b_\kappa$ such that the subspace $\prod b \setminus M$ of $\prod b$ is homeomorphic to ${}^\kappa\kappa$. ◁

Proof. First, we recursively define a function $\varphi : {}^{<\kappa}\kappa \to \prod_{<\kappa} b$. Let $\varphi(?) = ?$. Given $t \in {}^{<\kappa}\kappa$ such that $\varphi(t)$ has been defined, let $A = \{s_\alpha \mid \alpha \in \kappa\} \subseteq \prod_{<\kappa} b$ be an antichain of size κ with $\varphi(t) \subseteq s_\alpha$, such that A is maximal with this property, and send $\varphi : t^\frown\langle\alpha\rangle \mapsto s_\alpha$. If $\gamma = \mathrm{dom}(t)$ is limit and $\varphi(t \quad \alpha)$ is defined for each $\alpha < \gamma$, we have by construction that

$$\bigcup_{\alpha\in\gamma} \varphi(t \quad \alpha) \in \prod_{<\kappa} b.$$

Therefore we set $\varphi(t) = \bigcup_{\alpha\in\gamma} \varphi(t \quad \alpha)$.

Note that $\varphi[{}^\alpha\kappa]$ forms a maximal antichain in $\prod_{<\kappa} b$: if α is the least ordinal such that $\varphi[{}^\alpha\kappa]$ is not maximal, let $s \notin \varphi[{}^\alpha\kappa]$ be such that $\varphi[{}^\alpha\kappa] \cup \{s\}$ is an antichain, then for any $\xi < \alpha$ there is $s_\xi \subseteq s$ such that $s_\xi \in \varphi[{}^\xi\kappa]$, but then $\bigcup_{\xi\in\alpha} s_\xi = s' \subseteq s$ and $s' \in \varphi[{}^\alpha\kappa]$.

We define $\Phi : {}^\kappa\kappa \to \prod b$ induced by φ as sending $f \mapsto \bigcup_{\alpha\in\kappa} \varphi(f \quad \alpha)$. Note that the range $\Phi[{}^\kappa\kappa]$ is κ-comeagre in $\prod b$, because for each $\alpha \in \kappa$ we can define

$$F_\alpha = \left\{ f \in \prod b \mid f \notin \bigcup_{t\in{}^\alpha\kappa} [\varphi(t)] \right\}.$$

Then F_α is nowhere dense, since $\varphi[{}^\alpha\kappa]$ is a maximal antichain, and $\Phi[{}^\kappa\kappa] = \prod b \setminus \bigcup_{\alpha\in\kappa} F_\alpha$, hence $M = \bigcup_{\alpha\in\kappa} F_\alpha$ is a κ-meagre set and we will see that $\Phi : {}^\kappa\kappa \rightarrowtail\!\!\!\twoheadrightarrow \prod b \setminus M$ is a homeomorphism.

It is clear that Φ is bijective. Note that Φ is open, since for any $t \in {}^{<\kappa}\kappa$ we have $\varphi(t) \in \prod_{<\kappa} b$, thus $\Phi[[t]] = [\varphi(t)] \setminus M$ is open in $\prod b \setminus M$. To see that Φ is continuous, let $s \in \prod_{<\kappa} b$ with $\mathrm{dom}(s) = \alpha$. If $t \in {}^\alpha\kappa$, then it follows by construction that $\alpha \subseteq \mathrm{dom}(\varphi(t))$. Let

$$T = \{t \in {}^\alpha\kappa \mid s \subseteq \varphi(t)\}.$$

Note that $f \in \bigcup_{t\in T}[t]$ if and only if $s \subseteq \varphi(f \quad \alpha)$ if and only if $\Phi(f) \in [s]$ if and only if $f \in \Phi^{-1}[[s]]$. □

Corollary 3.1.2

$\mathrm{cov}(\mathcal{M}_\kappa) = \mathrm{cov}(\mathcal{M}^b_\kappa)$, $\mathrm{non}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}^b_\kappa)$, $\mathrm{add}(\mathcal{M}_\kappa) = \mathrm{add}(\mathcal{M}^b_\kappa)$, $\mathrm{cof}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}^b_\kappa)$. ◁

Apart from the κ-meagre ideal, we will also consider the ideal $\mathcal{SN}_\kappa$ of κ-strong measure zero sets in this chapter. We will conclude this section by giving a definition of $\mathcal{SN}_\kappa$ that is equivalent to the one given in Section 2.2, and some terminology that will be helpful to us later.

Lemma 3.1.3 — *for ${}^\omega\omega$, cf. [Mil81, Theorem 2.3], attributed to [Rot41]*

The following are equivalent for a set $X \subseteq {}^\kappa 2$.

(1) $X \in \mathcal{SN}_\kappa$,

(2) For every $f \in {}^\kappa\kappa$ there exists a sequence $\bar{s} = \langle s_\alpha \mid \alpha \in \kappa\rangle$ with $s_\alpha \in {}^{f(\alpha)}2$ for each α such that $X \subseteq \bigcap_{\alpha_0\in\kappa} \bigcup_{\beta\in[\alpha_0,\kappa)} [s_\beta]$. ◁

[1] Compare with Cantor schemes, e.g. as described in [Kec95, Section 6A] for ${}^\omega\omega$.

Proof. Remember that (1) holds iff for every $f \in {}^\kappa\kappa$ there exists a sequence $\bar{s} = \langle s_\alpha \mid \alpha \in \kappa\rangle$ with $s_\alpha \in {}^{f(\alpha)}2$ for each α such that $X \subseteq \bigcup_{\alpha\in\kappa}[s_\alpha]$.

That (2) implies (1) is obvious.

To see that (1) implies (2), let $\pi : \kappa \times \kappa \rightarrowtail\!\!\!\!\twoheadrightarrow \kappa$ be a bijection and $f \in {}^\kappa\kappa$. We define $f_\xi \in {}^\kappa\kappa$ by $f_\xi(\alpha) = f(\pi(\xi, \alpha))$ and use (1) to find a sequence $\bar{s}^\xi = \langle s^\xi_\alpha \mid \alpha \in \kappa\rangle$ with $s^\xi_\alpha \in {}^{f_\xi(\alpha)}2$ such that $X \subseteq \bigcup_{\alpha\in\kappa}[s^\xi_\alpha]$. Given $\xi, \alpha \in \kappa$, we define $s_{\pi(\xi,\alpha)} = s^\xi_\alpha$ then $s_\beta \in {}^{f(\beta)}2$ for all $\beta \in \kappa$. It follows that $X \subseteq \bigcap_{\alpha_0\in\kappa}\bigcup_{\beta\in[\alpha_0,\kappa)}[s_\beta]$. □

Definition 3.1.4
If $\bar{s} = \langle s_\alpha \mid \alpha \in \kappa\rangle$ is a sequence such that $X \subseteq \bigcup_{\alpha\in\kappa}[s_\alpha]$, we say X is ***covered*** by $\bar{s}$. If also $X \subseteq \bigcap_{\alpha_0\in\kappa}\bigcup_{\beta\in[\alpha_0,\kappa)}[s_\beta]$, then we say X is ***cofinally covered*** by $\bar{s}$. ◁

3.2. Bounded Cardinal Characteristics

Each of the relations $\leq^*$, $=^\infty$, $\in^*$ and $\in^\infty$ can be restricted to $\prod b$. For the former two this is clear, but for the latter two we need to also restrict the set of h-slaloms to the space $\prod b$, thereby defining a (b, h)-***slalom*** as a function φ with domain κ such that $\varphi(\alpha) \subseteq b(\alpha)$ and $|\varphi(\alpha)| < h(\alpha)$ for each $\alpha \in \kappa$. The set of all (b, h)-slaloms will be denoted by $\mathrm{Loc}^{b,h}_\kappa$.

Definition 3.2.1
We define the following four relational systems and associated cardinals:

$$\begin{array}{lll} \mathbf{D}^b_\kappa = \langle \prod b, \prod b, \leq^*\rangle & \left\|\mathbf{D}^b_\kappa\right\| = \mathfrak{d}^b_\kappa(\leq^*) & \left\|\mathbf{D}^{b\perp}_\kappa\right\| = \mathfrak{b}^b_\kappa(\leq^*) \\ \mathbf{ED}^b_\kappa = \langle \prod b, \prod b, =^\infty\rangle & \left\|\mathbf{ED}^b_\kappa\right\| = \mathfrak{d}^b_\kappa(=^\infty) & \left\|\mathbf{ED}^{b\perp}_\kappa\right\| = \mathfrak{b}^b_\kappa(=^\infty) \\ \mathbf{L}^{b,h}_\kappa = \langle \prod b, \mathrm{Loc}^{b,h}_\kappa, \in^*\rangle & \left\|\mathbf{L}^{b,h}_\kappa\right\| = \mathfrak{d}^{b,h}_\kappa(\in^*) & \left\|\mathbf{L}^{b,h\perp}_\kappa\right\| = \mathfrak{b}^{b,h}_\kappa(\in^*) \\ \mathbf{AL}^{b,h}_\kappa = \left\langle \mathrm{Loc}^{b,h}_\kappa, \prod b, \ni^\infty\right\rangle & \left\|\mathbf{AL}^{b,h}_\kappa\right\| = \mathfrak{d}^{b,h}_\kappa(\ni^\infty) & \left\|\mathbf{AL}^{b,h\perp}_\kappa\right\| = \mathfrak{b}^{b,h}_\kappa(\ni^\infty) \end{array}$$
◁

As for names for these cardinal characteristics, we will use the prefix b or (b, h) to distinguish these cardinals from the same cardinals as defined on the entire higher Baire space ${}^\kappa\kappa$ in Definition 2.4.5. That is, $\mathfrak{d}^b_\kappa(\leq^*)$ is called the b-***dominating number***, $\mathfrak{b}^{b,h}_\kappa(\in^*)$ is called the (b, h)-***avoidance number***, etc.

Classical Analogues of $\mathbf{D}^b_\kappa$

The b-dominating and b-unbounding numbers are examples of cardinal characteristics that are of independent interest only on higher Baire spaces. We will briefly discuss how these cardinals would be defined on ${}^\omega\omega$, and why this results in uninteresting cardinal characteristics.

If $b \in {}^\omega\omega$ is any cofinally increasing function, and we define $\mathbf{D}^b = \langle \prod b, \prod b, \leq^*\rangle$, it is easy to see that $\left\|\mathbf{D}^b\right\| = 1$, since the function $f = b - 1$ will dominate all functions in $\prod b$: if $g \in \prod b$, then $g(n) < b(n)$ for all $n \in \omega$, hence $g \leq^* f$. Similarly, this function f is a bound for all functions in $\prod b$, so there does not exist an unbounded family of functions, which makes $\left\|\mathbf{D}^{b\perp}\right\|$ undefined.

Through a different lense[2], we may see $\mathbf{D}^b_\kappa$ as a generalisation of a different concept. Given $b \in {}^\kappa\kappa$ a cofinal increasing infinite cardinal function and $g \in \prod b$, we consider the *difference* $\delta = b - g$, in the sense that $g(\alpha) + \delta(\alpha) = b(\alpha)$ for the function $\delta \in {}^\kappa\kappa$. Then δ is itself a cofinal function in ${}^\kappa\kappa$. In fact, one easily sees that $\delta = b$, since $b(\alpha)$ is always an infinite cardinal. We will use the fact that δ is a cofinal function to define a relational system in the context of ${}^\omega\omega$.

Given a cofinal function $b \in {}^\omega\omega$, define $\overset{\infty}{\prod} b = \{g \in \prod b \mid b - g \text{ is an increasing cofinal function}\}$. Consider $\mathbf{D}^b_\infty = \langle \overset{\infty}{\prod} b, \overset{\infty}{\prod} b, \leq^* \rangle$ and let $\left\|\mathbf{D}^b_\infty\right\| = \mathfrak{d}^b(\leq^*)$ and $\left\|\mathbf{D}^{b\,\perp}_\infty\right\| = \mathfrak{b}^b(\leq^*)$, then we may see $\mathfrak{d}^b_\kappa(\leq^*)$ and $\mathfrak{b}^b_\kappa(\leq^*)$ as generalisations of $\mathfrak{d}^b(\leq^*)$ and $\mathfrak{b}^b(\leq^*)$.

Lemma 3.2.2

For any cofinal function $b \in {}^\omega\omega$, we have $\mathbf{D}^b_\infty \equiv \mathbf{D}$. ◁

Proof. Let $f \in {}^\omega\omega$, where we will assume without loss of generality that $f(0) = 0$ and that f is an increasing cofinal function. We will define a function g_f as $g_f(k) = b(k) - n$ for each $k \in [f(n), f(n+1))$ and $n \in \omega$. We can see that $g_f \in \overset{\infty}{\prod} b$, since f is increasing and cofinal: for any $n \in \omega$ and $k \geq f(n)$ we have $b(k) - g_f(k) \geq n$.

Suppose that $f_1 \leq^* f_2$ for some $f_1, f_2 \in {}^\omega\omega$, and assume again without loss of generality that f_1 and f_2 are increasing and cofinal. Then we will show that $g_{f_1} \leq^* g_{f_2}$. Let $n_0 \in \omega$ be such that $f_1(n) \leq f_2(n)$ for all $n \geq n_0$. For any $k \geq f_1(n_0)$, we have $k \in [f_1(n_1), f_1(n_1+1))$ for some $n_1 \geq n_0$. By choice of n_0, we have $f_1(n_1) \leq f_2(n_1)$ and $f_1(n_1+1) \leq f_2(n_1+1)$, so there is $n_2 \leq n_1$ such that $k \in [f_2(n_2), f_2(n_2+1))$. But then $g_{f_1}(k) = b(k) - n_1 \leq b(k) - n_2 = g_{f_2}(k)$, hence $g_{f_1} \leq^* g_{f_2}$.

Reversely, given $g \in \overset{\infty}{\prod} b$, we define a function f_g by $f_g(n) = \min\{k \in \omega \mid b(k) - g(k) \geq n\}$. If $k \geq f_g(n)$, then $b(k) - g(k) \geq n$ as well, because $g \in \overset{\infty}{\prod} b$ assumes that $b - g$ is increasing.

Let $g_1 \leq^* g_2$ for some $g_1, g_2 \in \overset{\infty}{\prod} b$ and $k_0 \in \omega$ such that $g_1(k) \leq g_2(k)$ for all $k \geq k_0$. If $k \geq k_0$, then $b(k) - g_2(k) \geq n$ implies that $b(k) - g_1(k) \geq n$, hence $f_{g_2}(n) \geq f_{g_1}(n)$. Therefore $f_{g_1} \leq^* f_{g_2}$.

In conclusion, $\mathbf{D}^b_\infty \preceq \mathbf{D}$ is witnessed by the Tukey connection $\rho_- : g \mapsto f_g$ and $\rho_+ : f \mapsto g_f$ defined as above, while $\mathbf{D} \preceq \mathbf{D}^b_\infty$ has the Tukey connection $\rho_- : f \mapsto g_f$ and $\rho_+ : g \mapsto f_g$. □

It follows that $\mathfrak{d}^b(\leq^*) = \mathfrak{d}(\leq^*)$ and $\mathfrak{b}^b(\leq^*) = \mathfrak{b}(\leq^*)$, hence the bounded variants do not give us new cardinal characteristics, or at least not when we consider the ${}^\omega\omega$-analogues given above.

3.3. Relations

In this section we will prove ZFC-results concerning the bounded cardinal characteristics defined in the previous section. We do this by defining Tukey connections.

We will start with a few monotonicity results, then we look at the relation between the different flavours of cardinal characteristics and we conclude this section with infima and suprema of sets of cardinal characteristics indexed by the choice of b, which is related to the κ-strong measure zero ideal.

[2] This idea was proposed to me by Jörg Brendle in private communication.

To start, let us look at the b-dominating and b-unbounding numbers. We can easily see that only the cofinality of the values $b(\alpha)$ is important for these cardinals by the following lemma. Remember that $\mathrm{cf}(b) : \alpha \mapsto \mathrm{cf}(b(\alpha))$, as defined in Section 1.2.

Lemma 3.3.1
$\mathbf{D}_\kappa^b \equiv \mathbf{D}_\kappa^{\mathrm{cf}(b)}$. ◁

Proof. For each $\alpha \in \kappa$ let $\langle \beta_\xi^\alpha \mid \xi < \mathrm{cf}(b(\alpha))\rangle$ be a strictly increasing sequence of ordinals that is cofinal in $b(\alpha)$ and for any $\eta \in b(\alpha)$ let $\xi_\eta^\alpha = \min\{\xi \in \mathrm{cf}(b(\alpha)) \mid \eta \leq \beta_\xi^\alpha\}$

For $\mathbf{D}_\kappa^b \preceq \mathbf{D}_\kappa^{\mathrm{cf}(b)}$ let $\rho_-(f) : \alpha \mapsto \xi_{f(\alpha)}^\alpha$ and $\rho_+(g') : \alpha \mapsto \beta_{g'(\alpha)}^\alpha$.

For $\mathbf{D}_\kappa^{\mathrm{cf}(b)} \preceq \mathbf{D}_\kappa^b$ let $\rho_-(f') : \alpha \mapsto \beta_{f'(\alpha)}^\alpha$ and $\rho_+(g) : \alpha \mapsto \xi_{g(\alpha)}^\alpha$. □

The following lemma shows how taking a subsequence of b influences the cardinal characteristics under consideration.

Lemma 3.3.2
If $\langle \alpha_\xi \mid \xi \in \kappa\rangle \in {}^\kappa\kappa$ is a strictly increasing sequence and $b' : \xi \mapsto b(\alpha_\xi)$ and $h' : \xi \mapsto h(\alpha_\xi)$, then $\mathbf{D}_\kappa^{b'} \preceq \mathbf{D}_\kappa^b$ and $\mathbf{L}_\kappa^{b',h'} \preceq \mathbf{L}_\kappa^{b,h}$ and $\mathbf{AL}_\kappa^{b',h'} \preceq \mathbf{AL}_\kappa^{b,h}$. ◁

Proof. Each Tukey connection is similar, thus we will only give one.

Let $\rho_- : \prod b' \to \prod b$ send $f' \mapsto f$ where $f(\alpha_\xi) = f'(\xi)$ for each $\xi \in \kappa$ and arbitrary otherwise. Let $\rho_+ : \prod b \to \prod b'$ send $g \mapsto g'$ where $g'(\xi) = g(\alpha_\xi)$ for each $\xi \in \kappa$. It is easy to see that this is a Tukey connection for $\mathbf{D}_\kappa^{b'} \preceq \mathbf{D}_\kappa^b$. □

Another essential property relating two relational systems with different parameters to each other, is monotonicity with regards to the bounds b and h and the relation $\leq^*$. The proofs are elementary, and usually involve the identity functions as part of the Tukey connections, hence we omit them.

Lemma 3.3.3
Let $h \leq^* h'$ and $b \geq^* b'$, then $\mathbf{L}_\kappa^{b',h'} \preceq \mathbf{L}_\kappa^{b,h}$ and $\mathbf{AL}_\kappa^{b,h} \preceq \mathbf{AL}_\kappa^{b',h'}$.

Lastly, we mention that eventual difference is a special case of antilocalisation, where the function h that determines the size of the sets in the slaloms is as small as possible. Remember that $\overline{2} : \alpha \mapsto 2$, as defined in Section 1.2.

Lemma 3.3.4
$\mathbf{ED}_\kappa^b \equiv \mathbf{AL}_\kappa^{b,\overline{2}}$. ◁

Proof. Any slalom $\varphi \in \mathrm{Loc}_\kappa^{b,\overline{2}}$ is a sequence of singletons, and thus we can define $f_\varphi \in \prod b$ such that $\{f_\varphi(\alpha)\} = \varphi(\alpha)$ for all $\alpha \in \kappa$. It follows that $g \in^\infty \varphi$ if and only if $g =^\infty f_\varphi$. □

The following theorem gives an overview of relations between the cardinals we have discussed so far, for a fixed pair of parameters b, h.

Theorem 3.3.5
The following Tukey connections exist, where the relations marked by † require the additional assumption that $h \leq^* \mathrm{cf}(b)$.

$$\begin{array}{ccccc} \mathsf{AL}_\kappa^{b,h} & \preceq^\dagger & \mathsf{D}_\kappa^b & \preceq^\dagger & \mathsf{L}_\kappa^{b,h} \\ \rotatebox{90}{$\preceq$} & \rotatebox{45}{$\preceq$} & \rotatebox{90}{$\preceq$} & & \rotatebox{90}{$\preceq$} \\ \mathsf{ED}_\kappa^b & & \mathsf{D}_\kappa^{b\perp} & (\preceq^\dagger) & \mathsf{AL}_\kappa^{b,h\perp} \end{array}$$
◁

Proof. The relation between eventual difference and antilocalisation was already established with Lemma 3.3.4, and combined with Lemma 3.3.3 we get $\mathsf{ED}_\kappa^b \preceq \mathsf{AL}_\kappa^{b,h}$.

It is also easy to see for $f, g \in \prod b$, that if $g \leq^* f$, then $f+1 \leq^* g$ and $g =^\infty f+1$. This implies that there are Tukey connections $\mathsf{D}_\kappa^{b\perp} \preceq \mathsf{D}_\kappa^b$ and $\mathsf{ED}_\kappa^b \preceq \mathsf{D}_\kappa^b$.

Next, for $f \in \prod b$ and $\varphi \in \mathrm{Loc}_\kappa^{b,h}$, we have that $f \in^* \varphi$ implies $f \in^\infty \varphi$, and thus we get a Tukey connection $\mathsf{AL}_\kappa^{b,h\perp} \preceq \mathsf{L}_\kappa^{b,h}$.

Finally we will show that $\mathsf{AL}_\kappa^{b,h} \preceq \mathsf{D}_\kappa^b \preceq \mathsf{L}_\kappa^{b,h}$ as long as we assume that $h \leq^* \mathrm{cf}(b)$. Note that this also implies $\mathsf{AL}_\kappa^{b,h\perp} \succeq \mathsf{D}_\kappa^{b\perp}$ by duality.

We define $\rho_-^{\mathrm{AL}} : \mathrm{Loc}_\kappa^{b,h} \to \prod b$ and $\rho_+^{\mathrm{AL}} : \prod b \to \prod b$ as a Tukey connection for $\mathsf{AL}_\kappa^{b,h} \preceq \mathsf{D}_\kappa^b$, and $\rho_-^{\mathrm{L}} : \prod b \to \prod b$ and $\rho_+^{\mathrm{L}} : \mathrm{Loc}_\kappa^{b,h} \to \prod b$ as a Tukey connection for $\mathsf{D}_\kappa^b \preceq \mathsf{L}_\kappa^{b,h}$.

We will have $\rho_-^{\mathrm{AL}} = \rho_+^{\mathrm{L}}$ sending $\varphi \in \mathrm{Loc}_\kappa^{b,h}$ to $g \in \prod b$ where $g(\alpha) = \sup(\varphi(\alpha))$ if $\sup(\varphi(\alpha)) < b(\alpha)$ and arbitrary otherwise. Since $h \leq^* \mathrm{cf}(b)$, we see that $\varphi(\alpha)$ is not cofinal in $b(\alpha)$ for almost all α, so $g(\alpha) = \sup(\varphi(\alpha))$ for almost all $\alpha \in \kappa$. We let ρ_-^{L} be the identity function, and $\rho_+^{\mathrm{AL}} : f \mapsto f+1$.

If $f \in \prod b$ and $\varphi \in \mathrm{Loc}_\kappa^{b,h}$, let $g = \rho_-^{\mathrm{AL}}(\varphi) = \rho_+^{\mathrm{L}}(\varphi)$ and $f+1 = \rho_+^{\mathrm{AL}}(f)$. Then $f \in^* \varphi$ implies $f(\alpha) \leq \sup(\varphi(\alpha)) = g(\alpha)$ for almost all $\alpha \in \kappa$, hence $f \leq^* g$. On the other hand, if $g \leq^* f$, then $g <^* f+1$, hence $\sup(\varphi(\alpha)) < f(\alpha)+1$ for almost all $\alpha \in \kappa$, implying $f+1 \in^\infty \varphi$. □

In summary, we can draw the cardinal characteristics related to these relational systems in the diagram below, where the dashed lines require that $h \leq^* \mathrm{cf}(b)$. Note that $\mathfrak{b}_\kappa^b(=^\infty) \leq \mathrm{non}(\mathcal{M}_\kappa)$ and $\mathrm{cov}(\mathcal{M}_\kappa) \leq \mathfrak{d}_\kappa^b(=^\infty)$ follow from Lemmas 3.3.3 and 3.3.4 and Fact 2.5.4.

We saw that $\mathsf{AL}_\kappa^{b,h} \preceq \mathsf{D}_\kappa^b \preceq \mathsf{L}_\kappa^{b,h}$ if $h \leq^* \mathrm{cf}(b)$, and in Theorem 3.3.6 we showed that reversely $\mathsf{L}_\kappa^{b,h} \preceq \mathsf{D}_\kappa^b$ if $h =^* b$. Under the same assumption, we can also prove that $\mathsf{D}_\kappa^b \preceq \mathsf{AL}_\kappa^{b,h}$. Therefore, the dashed arrows in the above diagram collapse to become equalities in the case where $b =^* \mathrm{cf}(b) =^* h$.

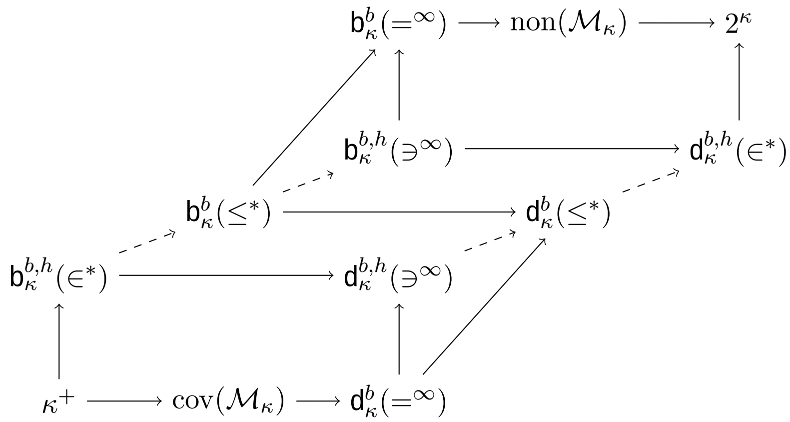


Figure 3.1: Diagram of the relations between cardinals on bounded spaces

Theorem 3.3.6
If $b =^* h$, then $\mathsf{L}_\kappa^{b,h} \preceq \mathsf{D}_\kappa^b \preceq \mathsf{AL}_\kappa^{b,h}$. ◁

Proof. We define $\rho_-^{\mathrm{L}}, \rho_+^{\mathrm{AL}} : \prod b \to \prod b$ and $\rho_+^{\mathrm{L}}, \rho_-^{\mathrm{AL}} : \prod b \to \mathrm{Loc}_\kappa^{b,h}$, so that $(\rho_-^{\mathrm{L}}, \rho_+^{\mathrm{L}})$ forms a Tukey connection for $\mathsf{L}_\kappa^{b,h} \preceq \mathsf{D}_\kappa^b$ and $(\rho_-^{\mathrm{AL}}, \rho_+^{\mathrm{AL}})$ forms a Tukey connection for $\mathsf{D}_\kappa^b \preceq \mathsf{AL}_\kappa^{b,h}$.

Let ρ_+^{AL} be the identity function and let $\rho_-^{\mathrm{L}} : f \mapsto f+1$. We will have $\rho_+^{\mathrm{L}} = \rho_-^{\mathrm{AL}}$ be the function sending $g \in \prod b$ to $\varphi \in \mathrm{Loc}_\kappa^{b,h}$ where $\varphi(\alpha) = g(\alpha)$ whenever $h(\alpha) = b(\alpha)$ and arbitrary otherwise. This is well-defined, since $|g(\alpha)| < b(\alpha) = h(\alpha)$ for all $\alpha \in \kappa$ on which φ is not arbitrary.

It is easy to check that these are Tukey connections. □

The relation $\mathsf{ED}_\kappa^b \preceq \mathsf{AL}_\kappa^{b,h}$ can also be reversed for certain choices of h and b, by the following theorem.

Theorem 3.3.7
Let b be increasing (not strictly) and $\langle I_\alpha \mid \alpha < \kappa \rangle$ be an interval partition of κ with $|I_\alpha| = h(\alpha)$ for each $\alpha \in \kappa$ such that $b(\alpha) = b(\xi) = b(\alpha)^{h(\alpha)}$ for all $\xi \in I_\alpha$ and $\alpha \in \kappa$, then $\mathsf{ED}_\kappa^b \equiv \mathsf{AL}_\kappa^{b,h}$. ◁

Proof. We already know $\mathsf{ED}_\kappa^b \preceq \mathsf{AL}_\kappa^{b,h}$ from Theorem 3.3.5, thus we show that $\mathsf{AL}_\kappa^{b,h} \preceq \mathsf{ED}_\kappa^b$. We do this by giving a Tukey connection $\rho_- : \mathrm{Loc}_\kappa^{b,h} \to \prod b$ and $\rho_+ : \prod b \to \prod b$. For each $\alpha \in \kappa$ let $\pi_\alpha : b(\alpha) \to {}^{I_\alpha}b(\alpha)$ be a bijection, which exist by $b(\alpha)^{h(\alpha)} = b(\alpha)$.

Given $\varphi \in \mathrm{Loc}_\kappa^{b,h}$, let $\lambda_\alpha = |\varphi(\alpha)|$ and enumerate each $\varphi(\alpha) = \{x_\xi^\alpha \in b(\alpha) \mid \xi \in \lambda_\alpha\}$. We define $g_\xi^\alpha = \pi_\alpha(x_\xi^\alpha) \in {}^{I_\alpha}b(\alpha)$ for all $\alpha \in \kappa$ and $\xi \in \lambda_\alpha$. Fix $\iota_\alpha = \min(I_\alpha)$, then for every $\xi \in \lambda_\alpha$ we see that $\iota_\alpha + \xi \in I_\alpha$, since $\lambda_\alpha < h(\alpha) = |I_\alpha|$. Let $g \in \prod b$ be a function such that $g(\iota_\alpha + \xi) = g_\xi^\alpha(\iota_\alpha + \xi)$ for each $\alpha \in \kappa$ and $\xi \in \lambda_\alpha$. This is well defined, because $b(\alpha) = b(\xi)$ for all $\xi \in I_\alpha$. We let $\rho_-(\varphi) = g$.

We define $\rho_+(f) \in \prod b$ sending $\alpha \mapsto \pi_\alpha^{-1}(f \restriction I_\alpha)$.

Now suppose that $\varphi \in \mathrm{Loc}_\kappa^{b,h}$ and $f \in \prod b$ and let $g = \rho_-(\varphi)$ and $f' = \rho_+(f)$. Suppose that $f' \in^\infty \varphi$ and $\alpha_0 \in \kappa$, then there is $\alpha > \alpha_0$ such that $f'(\alpha) \in \varphi(\alpha)$, so pick $\xi \in \lambda_\alpha$ such that

$x_\xi^\alpha = f'(\alpha)$. Then we see

$$g_\xi^\alpha = \pi_\alpha(x_\xi^\alpha) = \pi_\alpha(f'(\alpha)) = \pi_\alpha(\pi_\alpha^{-1}(f \restriction I_\alpha)) = f \restriction I_\alpha.$$

In particular $f(\iota_\alpha + \xi) = g_\xi^\alpha(\iota_\alpha + \xi) = g(\iota_\alpha + \xi)$, and since $\iota_\alpha + \xi > \alpha_0$ and α_0 was arbitrary, it follows that $f =^\infty g$. □

Note that the proof does not require the assumption that $b(\alpha) < \kappa$ for $\alpha \in \kappa$. Indeed, the above theorem holds even if we let $b : \kappa \to \{\kappa\}$, since κ is inaccessible. It follows that the cardinal characteristics $\mathsf{d}_\kappa^h(\ni^\infty)$ and $\mathsf{b}_\kappa^h(\ni^\infty)$ (from the unbounded space) do not depend on the choice of $h \in {}^\kappa\kappa$, and can be related to the κ-meagre ideal as a consequence of Fact 2.5.4.

Corollary 3.3.8
$\mathsf{d}_\kappa^h(\ni^\infty) = \operatorname{cov}(\mathcal{M}_\kappa)$ and $\mathsf{b}_\kappa^h(\ni^\infty) = \operatorname{non}(\mathcal{M}_\kappa)$ for any choice of $h \in {}^\kappa\kappa$. ◁

We will conclude this subsection with a relationship between antilocalisation and localisation for different parameters.

Theorem 3.3.9 — *cf. [KM22, Lemma 2.6] for ${}^\omega\omega$*
Let $h \cdot h' <^* b$ and $b^g \leq^* b'$ for all $g < h$, then there exists a Tukey connection $\mathsf{AL}_\kappa^{b,h} \preceq \mathsf{L}_\kappa^{b',h'}$. ◁

Proof. Let $\alpha_0 \in \kappa$ be large enough such that $h(\alpha) \cdot h'(\alpha) < b(\alpha)$ and $b(\alpha)^{<h(\alpha)} \leq b'(\alpha)$ for all $\alpha \geq \alpha_0$. For each $\alpha \geq \alpha_0$, we fix an injection $\iota_\alpha : [b(\alpha)]^{<h(\alpha)} \rightarrowtail b'(\alpha)$.

Given $\varphi \in \mathrm{Loc}_\kappa^{b,h}$, let $\rho_-(\varphi) = f' \in \prod b'$ map $\alpha \mapsto \iota_\alpha(\varphi(\alpha))$ for any $\alpha \geq \alpha_0$, and arbitrary otherwise. Given $\varphi' \in \mathrm{Loc}_\kappa^{b',h'}$, let $\rho_+(\varphi') = f \in \prod b$ have $f(\alpha) \in b(\alpha) \setminus \bigcup_{\xi \in \varphi'(\alpha) \cap \operatorname{ran}(\iota_\alpha)} \iota_\alpha^{-1}(\xi)$ for any $\alpha \geq \alpha_0$, and arbitrary otherwise. Note that this is well-defined, since $|\varphi'(\alpha)| < h'(\alpha)$ and $|\iota_\alpha^{-1}(\xi)| < h(\alpha)$, therefore $|\bigcup_{\xi \in \varphi'(\alpha) \cap \operatorname{ran}(\iota_\alpha)} \iota_\alpha^{-1}(\xi)| \leq h(\alpha) \cdot h'(\alpha) < b(\alpha)$.

If $\alpha \geq \alpha_0$ and $f'(\alpha) \in \varphi'(\alpha)$, then $\varphi(\alpha) = \iota_\alpha^{-1}(f'(\alpha)) \subseteq \bigcup_{\xi \in \varphi'(\alpha) \cap \operatorname{ran}(\iota_\alpha)} \iota_\alpha^{-1}(\xi)$, so $f(\alpha) \notin \varphi(\alpha)$. Therefore $f' \in^* \varphi'$ implies $f \in^\infty \varphi$. □

Infima and Suprema

Remember that we have the following monotonicity results from Lemma 3.3.3. Let $b \leq b'$. Then:

$$\mathsf{b}_\kappa^{b',h}(\in^*) \leq \mathsf{b}_\kappa^{b,h}(\in^*) \qquad \mathsf{d}_\kappa^{b,h}(\in^*) \leq \mathsf{d}_\kappa^{b',h}(\in^*)$$
$$\mathsf{b}_\kappa^{b,h}(\ni^\infty) \leq \mathsf{b}_\kappa^{b',h}(\ni^\infty) \qquad \mathsf{d}_\kappa^{b',h}(\ni^\infty) \leq \mathsf{d}_\kappa^{b,h}(\ni^\infty)$$

This motivates the definition of the following cardinal characteristics.

Definition 3.3.10
Given $h \in {}^\kappa\kappa$, let:

$$\mathsf{inf}_\kappa^h(\in^*) = \inf\{\mathsf{b}_\kappa^{b,h}(\in^*) \mid b \in {}^\kappa\kappa\} \qquad \mathsf{sup}_\kappa^h(\in^*) = \sup\{\mathsf{d}_\kappa^{b,h}(\in^*) \mid b \in {}^\kappa\kappa\}$$
$$\mathsf{inf}_\kappa^h(\ni^\infty) = \inf\{\mathsf{d}_\kappa^{b,h}(\ni^\infty) \mid b \in {}^\kappa\kappa\} \qquad \mathsf{sup}_\kappa^h(\ni^\infty) = \sup\{\mathsf{b}_\kappa^{b,h}(\ni^\infty) \mid b \in {}^\kappa\kappa\}$$
$$\mathsf{inf}_\kappa(=^\infty) = \inf\{\mathsf{d}_\kappa^{b}(=^\infty) \mid b \in {}^\kappa\kappa\} \qquad \mathsf{sup}_\kappa(=^\infty) = \sup\{\mathsf{b}_\kappa^{b}(=^\infty) \mid b \in {}^\kappa\kappa\}$$
◁

Note that for $\in^*$ we take infima over the $\mathfrak{b}$-side cardinals, while for $\ni^\infty$ and $=^\infty$ we take infima over the $\mathfrak{d}$-side cardinals, and vice versa for the suprema. Note also that $\mathsf{inf}_\kappa(=^\infty) = \mathsf{inf}^{\overline{2}}_\kappa(\ni^\infty)$ and $\mathsf{sup}_\kappa(=^\infty) = \mathsf{sup}^{\overline{2}}_\kappa(\ni^\infty)$ by Lemma 3.3.4.

In [CM19] it is proved that the parameter h does not matter if we work in ${}^\omega\omega$. This is based on two claims which we repeat below.

Fact 3.3.11 — *[CM19, Claim 3.10]*
For any $b, h, h' \in {}^\omega\omega$ such that $h \leq h'$ there exists $b' \in {}^\omega\omega$ such that $\mathfrak{d}^{b,h}_\omega(\in^*) \leq \mathfrak{d}^{b',h'}_\omega(\in^*)$ and $\mathfrak{b}^{b',h'}_\omega(\in^*) \leq \mathfrak{b}^{b,h}_\omega(\in^*)$.

Fact 3.3.12 — *[CM19, Claim 3.11]*
For any $b', h \in {}^\omega\omega$ such that $\overline{2} \leq h$ there exists $b \in {}^\omega\omega$ such that $\mathfrak{d}^{b,h}_\omega(\ni^\infty) \leq \mathfrak{d}^{b'}_\omega(=^\infty)$ and $\mathfrak{b}^{b'}_\omega(=^\infty) \leq \mathfrak{b}^{b,h}_\omega(\ni^\infty)$.

In the higher context, Fact 3.3.11 can only be partially generalised, and indeed we will see that differences in the parameter h can lead to consistently different cardinals. Meanwhile Fact 3.3.12 is completely generalisable and we will look at this first. One could compare the next lemma to Theorem 3.3.7.

Lemma 3.3.13
For any $b', h \in {}^\kappa\kappa$ such that $\overline{2} \leq h$ there exists $b \in {}^\kappa\kappa$ such that $\mathit{AL}^{b,h}_\kappa \preceq \mathit{ED}^{b'}_\kappa$. $\triangleleft$

Proof. Let $\langle I_\alpha \mid \alpha \in \kappa\rangle$ be the (unique) interval partition of κ such that $I_\alpha \subseteq \min(I_\beta)$ for $\alpha < \beta$ and $|I_\alpha| = h(\alpha)$ for all $\alpha \in \kappa$. We define $b(\alpha) = \left|\prod_{\xi\in I_\alpha} b'(\xi)\right|$ and let $\pi_\alpha : b(\alpha) \rightarrowtail\!\!\!\!\rightarrow \prod_{\xi\in I_\alpha} b'(\xi)$ be a bijection for each $\alpha \in \kappa$.

For each $\varphi \in \mathrm{Loc}^{b,h}_\kappa$, let $\rho_-(\varphi) = g_\varphi \in \prod b'$ be defined as follows. For each $\alpha \in \kappa$ we take some surjection $\iota^\varphi_\alpha : I_\alpha \twoheadrightarrow \varphi(\alpha)$. Given $\xi \in I_\alpha$, let $\beta = \iota^\varphi_\alpha(\xi) \in \varphi(\alpha)$, then we define $g_\varphi(\xi) = \pi_\alpha(\beta)(\xi)$.

For $f' \in \prod b'$, we let $\rho_+(f') = f \in \prod b$ be given by $f(\alpha) = \pi^{-1}_\alpha(f' \restriction I_\alpha)$.

Now (ρ_-, ρ_+) forms a Tukey connection. If $\rho_+(f') = f \in^\infty \varphi$, let $\alpha \in \kappa$ be arbitrarily large such that $f(\alpha) \in \varphi(\alpha)$. Take $\xi \in I_\alpha$ such that $f(\alpha) = \iota^\varphi_\alpha(\xi)$, then $g_\varphi(\xi) = \pi_\alpha(f(\alpha))(\xi) = f'(\xi)$, thus since α is arbitrarily large, we see that $f' =^\infty g_\varphi$. $\square$

Corollary 3.3.14
$\mathsf{inf}^h_\kappa(\ni^\infty) = \mathsf{inf}_\kappa(=^\infty)$ and $\mathsf{sup}^h_\kappa(\ni^\infty) = \mathsf{sup}_\kappa(=^\infty)$ for each $h \in {}^\kappa\kappa$. $\triangleleft$

Proof. By Lemma 3.3.3 we see that $\mathsf{inf}_\kappa(=^\infty) \leq \mathsf{inf}^h_\kappa(\ni^\infty)$ and $\mathsf{sup}_\kappa(=^\infty) \geq \mathsf{sup}^h_\kappa(\ni^\infty)$.

By Lemma 3.3.13, for any $b' \in {}^\kappa\kappa$ we can find some $b \in {}^\kappa\kappa$ such that $\mathfrak{d}^{b,h}_\kappa(\ni^\infty) \leq \mathfrak{d}^{b',\overline{2}}_\kappa(\ni^\infty)$, thus $\mathsf{inf}_\kappa(=^\infty) \geq \mathsf{inf}^h_\kappa(\ni^\infty)$. Similar for $\mathsf{sup}_\kappa(=^\infty) \leq \mathsf{sup}^h_\kappa(\ni^\infty)$. $\square$

The equality between $\mathsf{inf}_\omega(=^\infty)$ and $\mathrm{non}(\mathcal{SN})$ was first proved by Rothberger [Rot41]. Miller [Mil81, p.98 Remark (4)] used this result to remark that $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}_\omega(\leq^*), \mathsf{inf}_\omega(=^\infty)\}$. One could also show the dual result $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}_\omega(\leq^*), \mathsf{sup}_\omega(=^\infty)\}$. We show that each of these three claims generalises to higher context.

Instead of proving $\mathrm{non}(\mathcal{SN}_\kappa) = \mathsf{inf}_\kappa(=^\infty)$ directly, we show one direction by use of a Tukey connection. Let $\mathcal{X}_0 = \{(g, b) \in {}^\kappa\kappa \times {}^\kappa\kappa \mid g < b\}$ and $\mathcal{X}_1 = \{\tau : {}^\kappa\kappa \to {}^\kappa\kappa \mid \forall b \in {}^\kappa\kappa(\tau(b) < b)\}$, and define a relation $J_\infty \subseteq \mathcal{X}_0 \times \mathcal{X}_1$ by $(f, b)\ J_\infty\ \tau$ iff $f =^\infty \tau(b)$. We use this to define the relational system $\mathbf{J}_\infty$, which is equivalent to the categorical product $\bigotimes_{b\in{}^\kappa\kappa} \mathbf{ED}_\kappa^{b\perp}$. We also define a relational system whose norms are the covering and uniformity numbers of the κ-strong measure zero ideal.

$$\mathbf{J}_\infty = \langle \mathcal{X}_0, \mathcal{X}_1, J_\infty\rangle \qquad \|\mathbf{J}_\infty\| = \mathsf{sup}_\kappa(=^\infty) \qquad \left\|\mathbf{J}_\infty^\perp\right\| = \mathsf{inf}_\kappa(=^\infty)$$

$$\mathbf{C}_{\mathcal{SN}_\kappa} = \langle {}^\kappa 2, \mathcal{SN}_\kappa, \in\rangle \qquad \|\mathbf{C}_{\mathcal{SN}_\kappa}\| = \mathrm{cov}(\mathcal{SN}_\kappa) \qquad \left\|\mathbf{C}_{\mathcal{SN}_\kappa}^\perp\right\| = \mathrm{non}(\mathcal{SN}_\kappa)$$

The following Tukey connection proves $\mathsf{inf}_\kappa(=^\infty) \geq \mathrm{non}(\mathcal{SN}_\kappa)$ and $\mathsf{sup}_\kappa(=^\infty) \leq \mathrm{cov}(\mathcal{SN}_\kappa)$.

Theorem 3.3.15
$\mathbf{C}_{\mathcal{SN}_\kappa} \succeq \mathbf{J}_\infty$. ◁

Proof. We describe $\rho_- : \mathcal{X}_0 \to {}^\kappa 2$ and $\rho_+ : \mathcal{SN}_\kappa \to \mathcal{X}_1$.

For each $b \in {}^\kappa\kappa$ let $\beta^b_\alpha \in \kappa$ be minimal such that there exists an injection $\iota^b_\alpha : b(\alpha) \rightarrowtail {}^{\beta^b_\alpha}2$ and let γ^b_α be the ordinal sum $\sum_{\xi<\alpha} \beta^b_\xi$.

For any $(f, b) \in \mathcal{X}_0$, we define $\rho_-(f, b) = f' \in {}^\kappa 2$ piecewise by:

$$f' \quad [\gamma^b_\alpha, \gamma^b_{\alpha+1}) : \gamma^b_\alpha + \xi \mapsto \iota^b_\alpha(f(\alpha))(\xi).$$

Given $X \in \mathcal{SN}_\kappa$ and $b \in {}^\kappa\kappa$, we can find $\bar{s}^b = \langle s^b_\alpha \in {}^{\gamma^b_{\alpha+1}}2 \mid \alpha \in \kappa\rangle$ such that X is cofinally covered by $\bar{s}^b$. Define $t^b_\alpha \in {}^{\beta^b_\alpha}2$ by $t^b_\alpha : \xi \mapsto s^b_\alpha(\gamma^b_\alpha + \xi)$. Let $\rho_+(X) = \tau$, where $\tau(b) : \alpha \mapsto (\iota^b_\alpha)^{-1}(t^b_\alpha)$ if this is defined and arbitrary otherwise.

Given $(f, b) \in \mathcal{X}_0$ with $\rho_-(f, b) = f'$ and $X \in \mathcal{SN}_\kappa$ with $\rho_+(X) = \tau$, suppose that $f' \in X$, then since X is cofinally covered by $\bar{s}^b$, there are cofinally many α such that $f' \in [s^b_\alpha]$, hence for such α we have $f' \quad [\gamma^b_\alpha, \gamma^b_{\alpha+1}) = s^b_\alpha \quad [\gamma^b_\alpha, \gamma^b_{\alpha+1})$. But then $t^b_\alpha = \iota^b_\alpha(f(\alpha))$, and thus $\tau(b)(\alpha) = f(\alpha)$. Thus $f =^\infty \tau(b)$, or equivalently $(f, b)\ J_\infty\ \tau$. □

As said, $\mathrm{non}(\mathcal{SN}_\kappa)$ is actually equal to $\mathsf{inf}_\kappa(=^\infty)$. We prove the remaining direction below.

Theorem 3.3.16 — *cf. [Mil81, Theorem 2.3] for ${}^\omega\omega$*
$\mathrm{non}(\mathcal{SN}_\kappa) = \mathsf{inf}_\kappa(=^\infty)$. ◁

Proof. Let $\pi : {}^{<\kappa}2 \rightarrowtail\!\!\!\twoheadrightarrow \kappa$ be some fixed bijection and $X \notin \mathcal{SN}_\kappa$, then there exists $f \in {}^\kappa\kappa$ such that X is not cofinally covered for any $\bar{s} = \langle s_\alpha \mid \alpha \in \kappa\rangle$ with $s_\alpha \in {}^{f(\alpha)}2$. For each $x \in X$ let $g_x : \alpha \mapsto \pi(x \quad f(\alpha))$ and define $b : \alpha \mapsto \sup\left\{\pi(s) + 1 \mid s \in {}^{f(\alpha)}2\right\}$, then $g_x \in \prod b$. We set $D = \{g_x \mid x \in X\}$.

Given $h \in \prod b$, for each α define $s_\alpha = \pi^{-1}(h(\alpha))$ if $\mathrm{dom}(\pi^{-1}(h(\alpha))) = f(\alpha)$ and $s_\alpha \in {}^{f(\alpha)}2$ arbitrary otherwise. If $h(\alpha) = g_x(\alpha)$, then $s_\alpha = x \quad f(\alpha)$, hence $x \in [s_\alpha]$. Since X is not cofinally covered by $\bar{s} = \langle s_\alpha \mid \alpha \in \kappa\rangle$, there exists $x \in X$ such that $h =^\infty g_x$. Hence D forms a witness for $\mathsf{d}^b_\kappa(=^\infty)$. □

We cannot prove the dual $\mathrm{cov}(\mathcal{SN}_\kappa) \leq \mathsf{sup}_\kappa(=^\infty)$, since $\mathsf{sup}_\kappa(=^\infty) \leq \mathrm{non}(\mathcal{M}_\kappa) \leq \mathrm{cof}(\mathcal{M}_\kappa)$ and it is consistent that $\mathrm{cof}(\mathcal{M}_\kappa) < \mathrm{cov}(\mathcal{SN}_\kappa)$. The model for this is a generalisation of the Sacks model, which we will discuss in Section 4.4. We prove the consistency of this specific inequality in Theorem 4.4.9.

We will prove the connection between $\mathrm{add}(\mathcal{M}_\kappa)$ and $\mathsf{inf}_\kappa(=^\infty)$, and between $\mathrm{cof}(\mathcal{M}_\kappa)$ and $\mathsf{sup}_\kappa(=^\infty)$ directly, as it will be more similar to the proof of Theorem 3.3.19.

Theorem 3.3.17 — *cf. [Mil81, p.98 Remark (4)] for ${}^\omega\omega$*
$\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathsf{b}_\kappa(\leq^*), \mathsf{inf}_\kappa(=^\infty)\}$ and $\mathrm{cof}(\mathcal{M}_\kappa) = \max\{\mathsf{d}_\kappa(\leq^*), \mathsf{sup}_\kappa(=^\infty)\}$. ◁

Proof. Remember that by Fact 2.5.4 we have $\mathsf{d}_\kappa(=^\infty) = \mathrm{cov}(\mathcal{M}_\kappa)$ and $\mathsf{b}_\kappa(=^\infty) = \mathrm{non}(\mathcal{M}_\kappa)$, hence by Fact 2.5.9 $\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathsf{b}_\kappa(\leq^*), \mathsf{d}_\kappa(=^\infty)\}$ and $\mathrm{cof}(\mathcal{M}_\kappa) = \max\{\mathsf{d}_\kappa(\leq^*), \mathsf{b}_\kappa(=^\infty)\}$. Moreover, it is clear from Lemma 3.3.3 that $\mathsf{d}_\kappa(=^\infty) \leq \mathsf{inf}_\kappa(=^\infty)$ and $\mathsf{sup}_\kappa(=^\infty) \leq \mathsf{b}_\kappa(=^\infty)$.

Secondly, we prove that $\mathsf{d}_\kappa(=^\infty) < \mathsf{b}_\kappa(\leq^*)$ implies $\mathsf{d}_\kappa(=^\infty) = \mathsf{inf}_\kappa(=^\infty)$. Let $F \subseteq {}^\kappa\kappa$ be a witness for $|F| = \mathsf{d}_\kappa(=^\infty)$ and assume $\mathsf{d}_\kappa(=^\infty) < \mathsf{b}_\kappa(\leq^*)$. Since $|F| < \mathsf{b}_\kappa(\leq^*)$, there exists $b \in {}^\kappa\kappa$ such that $f <^* b$ for all $f \in F$. Let $f' : \alpha \mapsto f(\alpha)$ if $f(\alpha) \in b(\alpha)$ and $f' : \alpha \mapsto 0$ otherwise. Clearly $f' \in \prod b$ and for any $g \in {}^\kappa\kappa$ we have $f =^\infty g$ iff $f' =^\infty g$. Therefore $F' = \{f' \mid f \in F\} \subseteq \prod b$ is a witness for $\mathsf{d}^b_\kappa(=^\infty)$, thus we see:

$$\mathsf{d}_\kappa(=^\infty) = |F| \geq \mathsf{d}^b_\kappa(=^\infty) \geq \mathsf{inf}_\kappa(=^\infty) \geq \mathsf{d}_\kappa(=^\infty).$$

For the dual result, assume $\mathsf{d}_\kappa(\leq^*) < \mathsf{b}_\kappa(=^\infty)$ and towards contradiction let $\mathsf{sup}_\kappa(=^\infty) < \mathsf{b}_\kappa(=^\infty)$ as well. Let $D \subseteq {}^\kappa\kappa$ witness $|D| = \mathsf{d}_\kappa(\leq^*)$ and for each $b \in D$ choose a witness $F_b \subseteq \prod b$ for $|F_b| = \mathsf{b}^b_\kappa(=^\infty) \leq \mathsf{sup}_\kappa(=^\infty)$. Define $F = \bigcup_{b \in D} F_b$, then $|F| = \mathsf{d}_\kappa(\leq^*) \cdot \mathsf{sup}_\kappa(=^\infty) < \mathsf{b}_\kappa(=^\infty)$, thus there exists $g \in {}^\kappa\kappa$ such that $g =^\infty f$ for all $f \in F$. Let $b \in D$ be such that $g <^* b$ and let $g' : \alpha \mapsto g(\alpha)$ if $g(\alpha) < b(\alpha)$ and $g' : \alpha \mapsto 0$ otherwise, then $g' \in \prod b$ and $g' =^\infty f$ for all $f \in F_b$. But this contradicts that F_b witnesses $|F_b| = \mathsf{b}^b_\kappa(=^\infty)$.

Putting everything together, we have showed that:

$$\begin{aligned}
\mathrm{add}(\mathcal{M}_\kappa) &= \min\left\{\mathsf{b}_\kappa(\leq^*), \mathsf{d}_\kappa(=^\infty)\right\} = \min\left\{\mathsf{b}_\kappa(\leq^*), \mathsf{inf}_\kappa(=^\infty)\right\}, \\
\mathrm{cof}(\mathcal{M}_\kappa) &= \max\left\{\mathsf{d}_\kappa(\leq^*), \mathsf{b}_\kappa(=^\infty)\right\} = \max\left\{\mathsf{d}_\kappa(\leq^*), \mathsf{sup}_\kappa(=^\infty)\right\}.
\end{aligned}$$ □

We will now show a generalisation of Fact 3.3.11. It is not possible to prove the generalisation for every $b, h, h' \in {}^\kappa\kappa$ with $h \leq h'$, since we will see in the next section that it is consistent that $\mathsf{d}^{b,2^h}_\kappa(\in^*) = \mathsf{d}^{b',2^h}_\kappa(\in^*) < \mathsf{d}^{b,h}_\kappa(\in^*) = \mathsf{d}^{b',h}_\kappa(\in^*)$ for all $b' \geq b$.

Lemma 3.3.18
For any $b, h, h' \in {}^\kappa\kappa$ such that there exists a continuous strictly increasing sequence $\langle \beta_\alpha \mid \alpha \in \kappa \rangle$ with $h'(\alpha) \leq h(\xi)$ for all $\xi \geq \beta_\alpha$, there exists $b' \in {}^\kappa\kappa$ such that $\mathsf{L}_{b,h} \preceq \mathsf{L}_{b',h'}$. ◁

Proof. Let $I_\alpha = [\beta_\alpha, \beta_{\alpha+1})$ for each $\alpha \in \kappa$, where we assume without loss of generality that $\beta_0 = 0$. Since $\langle \beta_\xi \mid \xi \in \kappa \rangle$ is continuous, we see that $\langle I_\alpha \mid \alpha \in \kappa \rangle$ is an interval partition of κ. We define $b'(\alpha) = \left| \prod_{\xi \in I_\alpha} b(\xi) \right|$ and a bijection $\pi_\alpha : b'(\alpha) \rightarrowtail\!\!\!\!\rightarrow \prod_{\xi \in I_\alpha} b(\xi)$ for each $\alpha \in \kappa$.

$\rho_- : \prod b \to \prod b'$ is defined by $\rho_-(f) = f' : \alpha \mapsto \pi_\alpha^{-1}(f \quad I_\alpha)$. We define $\rho_+ : \mathrm{Loc}_\kappa^{b',h'} \to \mathrm{Loc}_\kappa^{b,h}$ by $\rho_+(\varphi') = \varphi : \xi \mapsto \{\pi_\alpha(x)(\xi) \mid x \in \varphi'(\alpha)\}$, where α is such that $\xi \in I_\alpha$. Note that this is well-defined, since $\xi \in I_\alpha$ implies $\xi \geq \beta_\alpha$, and thus $|\varphi'(\alpha)| < h'(\alpha) \leq h(\xi)$.

Suppose that $f \in \prod b$, $\varphi' \in \mathrm{Loc}_\kappa^{b',h'}$ and $\rho_-(f) = f'$, $\rho_+(\varphi') = \varphi$. Let $\alpha \in \kappa$ be such that $f'(\alpha) \in \varphi'(\alpha)$. For any $\xi \in I_\alpha$, we then see that $\pi_\alpha(f'(\alpha))(\xi) = f(\xi)$, thus $f(\xi) \in \varphi(\xi)$ for all $\xi \in I_\alpha$. Hence, if $f' \in^* \varphi'$, it follows that $f \in^* \varphi$. □

Classically analogous to Theorem 3.3.17, it can be shown that $\mathrm{add}(\mathcal{N}) = \min\{\mathfrak{b}_\omega(\leq^*), \mathfrak{inf}_\omega(\in^*)\}$ and $\mathrm{cof}(\mathcal{N}) = \max\{\mathfrak{d}_\omega(\leq^*), \mathfrak{sup}_\omega(\in^*)\}$. There is a generalisation of this result for ${}^\kappa\kappa$ as well, although we have to replace $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ by their combinatorial counterparts $\mathfrak{b}_\kappa^h(\in^*)$ and $\mathfrak{d}_\kappa^h(\in^*)$ (nota bene: these are the cardinals from the (unbounded) higher Baire space ${}^\kappa\kappa$, or equivalently we could regard the bound to be $b = \overline{\kappa}$). Moreover, since the parameter h is important, in the sense that differing h leads to different cardinals, we can only prove a parametrised version of this result.

Theorem 3.3.19 — *cf. [CM19, Lemma 3.12] for ${}^\omega\omega$*
$\mathfrak{b}_\kappa^h(\in^*) = \min\left\{\mathfrak{b}_\kappa(\leq^*), \mathfrak{inf}_\kappa^h(\in^*)\right\}$ and $\mathfrak{d}_\kappa^h(\in^*) = \max\left\{\mathfrak{d}_\kappa(\leq^*), \mathfrak{sup}_\kappa^h(\in^*)\right\}$ for any $h \in {}^\kappa\kappa$. ◁

Proof. That $\mathfrak{b}_\kappa^h(\in^*) \leq \min\left\{\mathfrak{b}_\kappa(\leq^*), \mathfrak{inf}_\kappa^h(\in^*)\right\}$ and $\mathfrak{d}_\kappa^h(\in^*) \geq \max\left\{\mathfrak{d}_\kappa(\leq^*), \mathfrak{sup}_\kappa^h(\in^*)\right\}$ are clear.

Let $F \subseteq {}^\kappa\kappa$ with $|F| < \min\left\{\mathfrak{b}_\kappa(\leq^*), \mathfrak{inf}_\kappa^h(\in^*)\right\}$, then there exists $b \in {}^\kappa\kappa$ such that $f <^* b$ for all $f \in F$. Let $F' = \{f' \mid f \in F\}$ where $f' : \alpha \mapsto f(\alpha)$ if $f(\alpha) < b(\alpha)$ and $f' : \alpha \mapsto 0$ otherwise, then $f =^* f' \in \prod b$. Finally, there exists $\varphi \in \mathrm{Loc}_\kappa^{b,h}$ such that $f' \in^* \varphi$ for all $f' \in F'$ by $|F'| < \mathfrak{inf}_\kappa^h(\in^*) \leq \mathfrak{b}_\kappa^{b,h}(\in^*)$. Then also $f \in^* \varphi$ for all $f \in F$, thus $|F| < \mathfrak{b}_\kappa^h(\in^*)$.

Let $D \subseteq {}^\kappa\kappa$ be a witness for $\mathfrak{d}_\kappa(\leq^*)$ with $|D| = \mathfrak{d}_\kappa(\leq^*)$. For each $b \in D$ choose a witness $\Phi_b \subseteq \mathrm{Loc}_\kappa^{b,h}$ for $\mathfrak{d}_\kappa^{b,h}(\in^*)$ with $|\Phi_b| = \mathfrak{d}_\kappa^{b,h}(\in^*)$. Let $\Phi = \bigcup_{b \in D} \Phi_b$, then $|\Phi| \leq \max\left\{\mathfrak{d}_\kappa(\leq^*), \mathfrak{sup}_\kappa^h(\in^*)\right\}$. If $f \in {}^\kappa\kappa$, then there is $b \in D$ such that $f <^* b$. Again, let $f' : \alpha \mapsto f(\alpha)$ if $f(\alpha) < b(\alpha)$ and $f' : \alpha \mapsto 0$ otherwise, then $f' \in \prod b$. Therefore, there exists $\varphi \in \Phi_b$ such that $f' \in^* \varphi$, and thus such that $f \in^* \varphi$. This shows that Φ is a witness for $\mathfrak{d}_\kappa^h(\in^*)$. □

3.4. Trivial Parameters

It is perhaps not very surprising that some choices of parameters b and h will result in the cardinal characteristics having trivial values. What we mean precisely with a cardinal characteristic $\mathfrak{x}$ having a "trivial" value, is that $\mathfrak{x}$ is undefined, or that it is provable in "$\mathsf{ZFC} + \kappa$ is inaccessible" that $\mathfrak{x} \leq \kappa^+$ or $\mathfrak{x} = 2^\kappa$.

It is perhaps not evident that the determination of b and h such that our cardinals are nontrivial, is itself not a trivial task. Indeed, this section contains several nontrivial open questions regarding the triviality of cardinal characteristics.

We will establish a general pattern that it is possible to give a complete characterisation of the cases in which the cardinals $\mathfrak{b}_\kappa^b(\leq^*)$, $\mathfrak{b}_\kappa^{b,h}(\in^*)$, $\mathfrak{b}_\kappa^{b,h}(\ni^\infty)$ and $\mathfrak{b}_\kappa^b(=^\infty)$ are trivial. For each of these families of cardinals we are able to formulate a trichotomy of the case where the cardinal is $< \kappa$, the case where it is exactly κ and the case where the cardinal is $> \kappa$. In the last case we can show that the cardinal is nontrivial, and we will give an independence proof in Chapter 4.

For the cardinals $\mathfrak{d}^b_\kappa(\leq^*)$, $\mathfrak{d}^{b,h}_\kappa(\in^*)$, $\mathfrak{d}^{b,h}_\kappa(\ni^\infty)$ and $\mathfrak{d}^b_\kappa(=^\infty)$, the natural conjecture is that these are trivial exactly when their $\mathfrak{b}$-duals are trivial, but this turns out to be hard to prove in each case. We will give partial results and some related problems. On the other hand, we can show that these $\mathfrak{d}$-side cardinals are nontrivial whenever the $\mathfrak{b}$-side cardinals are nontrivial by dual independence proofs, also given in Chapter 4.

Domination & Unboundedness

Let us start with $\mathfrak{d}^b_\kappa(\leq^*)$ and $\mathfrak{b}^b_\kappa(\leq^*)$. We saw in Lemma 3.3.1 that the value of these cardinals only depends on the cofinality of $b(\alpha)$. It is also quite immediate that we require $b \in {}^\kappa\kappa$ to be such that $b(\alpha)$ is an infinite cardinal for (almost) all $\alpha \in \kappa$ for the same reason that a direct ${}^\omega\omega$-analogue, as given at the end of Section 3.2, gives us trivial values:

Lemma 3.4.1
If $b(\alpha)$ is a successor ordinal for almost all α, then $\mathfrak{d}^b_\kappa(\leq^*) = 1$ and $\mathfrak{b}^b_\kappa(\leq^*)$ is undefined. ◁

Proof. Let $b(\alpha) = f(\alpha) + 1$ for almost all $\alpha \in \kappa$, then clearly f dominates all functions in $\prod b$, hence $\mathfrak{d}^b_\kappa(\leq^*) = 1$ and $\mathfrak{b}^b_\kappa(\leq^*)$ is undefined. □

The following lemma gives a complete characterisation of the functions b for which $\mathfrak{b}^b_\kappa(\leq^*)$ is trivial. Note that the cases (i), (ii) and (iii) form a trichotomy.

Theorem 3.4.2
For each regular cardinal $\lambda < \kappa$ let $D_\lambda = \{\alpha \in \kappa \mid \mathrm{cf}(b(\alpha)) = \lambda\}$.

(i) If there exists a least regular cardinal $\lambda < \kappa$ such that D_λ is cofinal in κ, then $\mathfrak{b}^b_\kappa(\leq^*) = \lambda$.
(ii) If D_λ is bounded for all regular $\lambda < \kappa$ and there exists a stationary set S such that for each $\xi \in S$ there exists $\alpha_\xi \geq \xi$ with $\mathrm{cf}(b(\alpha_\xi)) \leq \xi$, then $\mathfrak{b}^b_\kappa(\leq^*) = \kappa$.
(iii) If D_λ is bounded for all regular $\lambda < \kappa$ and there exists a club set C such that for each $\xi \in C$ we have $\mathrm{cf}(b(\alpha)) > \xi$ for all $\alpha \geq \xi$, then $\mathfrak{b}^b_\kappa(\leq^*) \geq \kappa^+$. ◁

Proof. (i) For each $\gamma \in D_\lambda$ let $\langle \delta^\alpha_\gamma \mid \alpha \in \lambda \rangle$ be an increasing cofinal sequence in γ. Let $f_\alpha \in \prod b$ be any function such that $f_\alpha(\gamma) = \delta^\alpha_\gamma$ for each $\gamma \in D_\lambda$, then we claim that $B = \{f_\alpha \mid \alpha \in \lambda\}$ is unbounded. Let $g \in \prod b$. By the pigeonhole principle there exists $\alpha \in \lambda$ such that $g(\gamma) < \delta^\alpha_\gamma$ for cofinally many $\gamma \in D_\lambda$, hence we see that $f_\alpha \not\leq^* g$.

On the other hand, if $|B| < \lambda$, then let α_0 be large enough such that $\mathrm{cf}(b(\alpha)) \geq \lambda$ for all $\alpha \geq \alpha_0$, then $|\{f(\alpha) \mid f \in B\}| < \mathrm{cf}(b(\alpha))$ for all $\alpha \geq \alpha_0$, thus we can pick $g \in \prod b$ such that $g(\alpha) = \sup\{f(\alpha) \mid f \in B\} < b(\alpha)$ for each $\alpha \geq \alpha_0$ to see that $f \leq^* g$ for all $f \in B$.

(ii) Since each D_λ is bounded, we may assume that $\alpha_\xi \neq \alpha_{\xi'}$ for all distinct $\xi, \xi' \in S$. For each $\xi \in S$, let $\{\beta^\eta_\xi \mid \eta \in \xi\}$ be a cofinal subset of $b(\alpha_\xi)$ (not necessarily increasing). Given $\alpha \in \kappa$ we define f_α such that $f_\alpha(\alpha_\xi) = \beta_\xi^\alpha$ for all $\xi \in S$ with $\alpha < \xi$ and arbitrary otherwise. We claim that $B = \{f_\alpha \mid \alpha \in \kappa\}$ is unbounded.

Let $g \in \prod b$, then for each $\xi \in S$ there is a minimal $\eta_\xi < \xi$ such that $g(\alpha_\xi) < \beta_\xi^{\eta_\xi} = f_{\eta_\xi}(\alpha_\xi)$, hence $g' : \xi \mapsto \eta_\xi$ is a regressive function on S. Therefore there is stationary $S' \subseteq S$ such that g' is constant on S', say with value η, then we see that $f_\eta \leq^* g$.

On the other hand, if $B \in [\prod b]^\mu$ with $\mu < \kappa$ and we take regular $\lambda > \mu$ such that D_λ is bounded, then we can argue as in case (i) to construct $g \in \prod b$ with $f \leq^* g$ for all $f \in B$.

(iii) Let $B \subseteq \prod b$ with $|B| = \kappa$ and enumerate B as $\{f_\eta \mid \eta \in \kappa\}$. Let $\langle \alpha_\xi \mid \xi \in \kappa \rangle$ be an increasing enumeration of C. We define $g(\alpha) = \bigcup_{\eta \in \xi} f_\eta(\alpha)$ for all $\alpha \in [\alpha_\xi, \alpha_{\xi+1})$ and $\xi \in \kappa$, then $g(\alpha)$ is the supremum of a sequence of length $\xi \leq \alpha_\xi < \mathrm{cf}(b(\alpha))$, hence $g \in \prod b$. Clearly $f_\xi \leq^* g$ for each $\xi \in \kappa$, so B is bounded. $\square$

Hence, we see that $\mathfrak{b}_\kappa^b(\leq^*)$ is trivial in cases (i) and (ii). We will later prove that $\kappa^+ < \mathfrak{b}_\kappa^b(\leq^*)$ is consistent in case (iii), but let us take a look at the dual $\mathfrak{d}_\kappa^b(\leq^*)$ first.

If $\mathfrak{d}_\kappa^b(\leq^*)$ behaves dually to $\mathfrak{b}_\kappa^b(\leq^*)$, then we expect $\mathfrak{d}_\kappa^b(\leq^*) < 2^\kappa$ to be inconsistent in cases (i) and (ii). Remember that we assume that b is increasing, hence by Lemma 3.3.1, we can reduce case (i) to the situation where b is a constant function with a regular cardinal as value. In other words, we have to study the dominating number as defined in the space ${}^\kappa\lambda$ where $\lambda < \kappa$ is a regular cardinal.

Note that even if we drop the assumption that b is increasing, Lemma 3.3.2 shows that the behaviour on the space ${}^\kappa\lambda$ is essentially the relevant part of the (in)consistency of $\mathfrak{d}_\kappa^b(\leq^*) < 2^\kappa$.

Dominating numbers in the space ${}^\kappa\lambda$ have been studied by many in the past. Brendle showed in the last section of [Bre22] (using different notation where the roles of κ and λ are reversed) that if $\lambda < \kappa$ and λ is regular uncountable, then $\mathfrak{d}_\kappa^{\bar\lambda}(\leq^*) < 2^\kappa$ is actually consistent. An example for a model where this holds, is the model resulting from adding κ^{++} many μ-Cohen reals over GCH, where $\mu < \lambda$. Because this also destroys the inaccessibility of κ, we cannot use this in our context.

The question whether $\mathfrak{d}_\kappa^{\bar\lambda}(\leq^*) < 2^\kappa$ is consistent with $\kappa \geq 2^{<\lambda}$ is mentioned as Question 16 in [Bre22]. Moreover, the special case where $\kappa = \omega_1$ and $\lambda = \omega$ is a famous open problem of Jech & Prikry [JP79] that is still unsolved almost half a century later. We will give a partial answer and prove that $\mathfrak{d}_\kappa^{\bar\lambda}(\leq^*) = 2^\kappa$ when κ is inaccessible.

Theorem 3.4.3
If $\lambda < \kappa$ is regular and $\mathrm{cf}(b(\alpha)) = \lambda$ for cofinally many $\alpha \in \kappa$, then $\mathfrak{d}_\kappa^b(\leq^*) = 2^\kappa$. $\triangleleft$

We will delay the essential part of the proof of this theorem to the next subsection, since it will be a corollary of Theorem 3.4.6, which states that $\mathfrak{d}_\kappa^{b,h}(\in^*) = 2^\kappa$ if $h(\alpha) = \lambda$ for cofinally many $\alpha \in \kappa$. This is related to $\mathfrak{d}_\kappa^b(\leq^*)$ through Theorem 3.3.6.

Proof of Theorem 3.4.3. Let $\lambda < \kappa$ be regular and $\mathrm{cf}(b(\alpha)) = \lambda$ for cofinally many $\alpha \in \kappa$. By Lemma 3.3.1 and Lemma 3.3.2 we may assume that $b(\alpha) = \lambda$ for all $\alpha \in \kappa$. Let $h = b$, then the conditions of Theorems 3.3.6 and 3.4.6 are satisfied, thus $2^\kappa = \mathsf{d}^{b,h}_\kappa(\in^*) \leq \mathsf{d}^{b}_\kappa(\leq^*) \leq 2^\kappa$. □

We currently do not know whether $\mathsf{d}^{b}_\kappa(\leq^*)$ also is trivial in case (ii) of Theorem 3.4.2, see also Question 3.5.1. We can however show that both $\mathsf{b}^{b}_\kappa(\leq^*)$ and $\mathsf{d}^{b}_\kappa(\leq^*)$ are nontrivial in case (iii), and we will give an independence proof separating each from their respective bound in Theorem 4.3.19.

Localisation & Avoidance

With localisation and avoidance cardinals, we have not only the parameter b, but also the parameter h giving the width of our slaloms. As with the dominating and unbounding numbers, there are certain choices of these parameters for which we have trivial cardinal characteristics.

We will motivate our assumption that $h \leq b$ with the following two lemmas:

Lemma 3.4.4
$b <^* h$ if and only if $\mathsf{d}^{b,h}_\kappa(\in^*) = 1$.
If $b <^* h$, then $\mathsf{b}^{b,h}_\kappa(\in^*)$ is undefined. ◁

Proof. If $b <^* h$, let $B = \{\alpha \in \kappa \mid b(\alpha) < h(\alpha)\}$, and choose some $\varphi \in \mathrm{Loc}^{b,h}_\kappa$ such that $\varphi(\alpha) = b(\alpha)$ for all $\alpha \in B$. Since almost all $\alpha \in \kappa$ are in B, we see that $f \in \prod b$ implies $f \in^* \varphi$. Hence φ localises the entirety of $\prod b$, making $\mathsf{d}^{b,h}_\kappa(\in^*) = 1$ and $\mathsf{b}^{b,h}_\kappa(\in^*)$ undefined.

Reversely, if there is a strictly increasing sequence $\langle \alpha_\xi \mid \xi \in \kappa\rangle$ with $h(\alpha_\xi) \leq b(\alpha_\xi)$ for all ξ, and $\varphi \in \mathrm{Loc}^{b,h}_\kappa$, then there is $\gamma_\xi \in b(\alpha_\xi) \setminus \varphi(\alpha_\xi)$ for each ξ since $|\varphi(\alpha_\xi)| < h(\alpha_\xi) \leq b(\alpha_\xi)$. Therefore, if $f \in \prod b$ is such that $f(\alpha_\xi) = \gamma_\xi$ for all $\xi \in \kappa$, then $f \notin^* \varphi$. □

Mirroring the situation of Theorem 3.4.2, we will give a complete characterisation of the cases in which $\mathsf{b}^{b,h}_\kappa(\in^*)$ is trivial.

Theorem 3.4.5
For each $\lambda < \kappa$ define $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$.

(i) If there exists a least cardinal $\lambda < \kappa$ such that D_λ is cofinal in κ, then $\mathsf{b}^{b,h}_\kappa(\in^*) = \lambda$.
(ii) If D_λ is bounded for all $\lambda < \kappa$ and h is continuous on a stationary set, then $\mathsf{b}^{b,h}_\kappa(\in^*) = \kappa$,
(iii) If D_λ is bounded for all $\lambda < \kappa$ and h is discontinuous on a club set, then $\kappa^+ \leq \mathsf{b}^{b,h}_\kappa(\in^*)$. ◁

Proof. (i) Assume that λ is minimal such that D_λ is cofinal. For each $\eta < \lambda$, define $f_\eta \in \prod b$ elementwise by $f_\eta : \alpha \mapsto \eta$ for all $\alpha \in D_\lambda$ and $f_\eta : \alpha \mapsto 0$ otherwise. If $\varphi \in \mathrm{Loc}^{b,h}_\kappa$ and $\alpha \in D_\lambda$, then $|\varphi(\alpha)| < h(\alpha) = \lambda$, so there exists $\eta < \lambda$ for which $\eta \notin \varphi(\alpha)$ for cofinally many $\alpha \in D_\lambda$ by the pigeonhole principle, hence $f_\eta \notin^* \varphi$. Therefore $\mathcal{F} = \{f_\eta \mid \eta < \lambda\}$ witnesses that $\mathsf{b}^{b,h}_\kappa(\in^*) \leq |\mathcal{F}| = \lambda$.

On the other hand, if $\mathcal{F} \subseteq \prod b$ and $|\mathcal{F}| < \lambda$, then by minimality of λ we see that $|\mathcal{F}| < h(\alpha)$ for almost all $\alpha \in \kappa$, thus we can choose $\varphi \in \mathrm{Loc}^{b,h}_\kappa$ such that $\varphi : \alpha \mapsto \{f(\alpha) \mid f \in \mathcal{F}\}$ whenever $|\mathcal{F}| < h(\alpha)$, then we see that $f \in^* \varphi$ for all $f \in \mathcal{F}$, proving that $|\mathcal{F}| < \mathsf{b}^{b,h}_\kappa(\in^*)$.

(ii) Since each D_λ is bounded, it follows that $\{\alpha \in \kappa \mid h(\alpha) \leq \lambda\}$ is bounded for every $\lambda < \kappa$. Let $\mathcal{F} \subseteq \prod b$ with $|\mathcal{F}| < \kappa$, and let $\beta \in \kappa$ be such that $h(\alpha) \leq |\mathcal{F}|$ implies $\alpha < \beta$. If $\varphi \in \mathrm{Loc}_\kappa^{b,h}$ and $\xi \geq \beta$, then $|\mathcal{F}| < h(\xi)$, hence we can define $\varphi(\xi) = \{f(\xi) \mid f \in \mathcal{F}\}$ for all $\xi \geq \beta$. It is clear that $f \in^* \varphi$ for all $f \in \mathcal{F}$, thus $|\mathcal{F}| < \mathsf{b}_\kappa^{b,h}(\in^*)$. Since $\mathcal{F}$ was arbitrary such that $|\mathcal{F}| < \kappa$, we see that $\kappa \leq \mathsf{b}_\kappa^{b,h}(\in^*)$.

To see $\kappa = \mathsf{b}_\kappa^{b,h}(\in^*)$, consider $\mathcal{F} = \{f_\eta \mid \eta \in \kappa\}$ where $f_\eta : \alpha \mapsto \eta$ when $\eta \in b(\alpha)$, and $f_\eta : \alpha \mapsto 0$ otherwise. Since we assume $h \leq^* b$ and h is cofinal, for every $\eta \in \kappa$ there is α_0 such that $b(\alpha) > \eta$ for all $\alpha \geq \alpha_0$, thus $f_\eta(\alpha) = \eta$ for almost all $\alpha \in \kappa$. Since h is continuous on a stationary set S_0, the set of fixed-points $S_1 = \{\alpha \in S_0 \mid h(\alpha) = \alpha\}$ is stationary, implying that $\alpha \mapsto |\varphi(\alpha)|$ is regressive on the stationary S_1. Consequently Fodor's lemma tells us that there exists a stationary set $S_2 \subseteq S_1$ and $\lambda \in \kappa$ such that $|\varphi(\alpha)| < \lambda$ for all $\alpha \in S_2$. If $f_\eta \in^* \varphi$ for all $\eta \in \lambda$, then $N_\eta = \{\alpha \in \kappa \mid \eta \notin \varphi(\alpha)\}$ is nonstationary for each $\eta \in \lambda$, so $\bigcup_{\eta \in \lambda} N_\eta$ is nonstationary, and thus $S_2 \setminus \bigcup_{\eta\in\lambda} N_\eta$ is stationary. However, for any $\alpha \in S_2 \setminus \bigcup_{\eta\in\lambda} N_\eta$ we have $\lambda \subseteq \varphi(\alpha)$, which contradicts that $|\varphi(\alpha)| < \lambda$ for $\alpha \in S_2$. Therefore $\mathcal{F}$ forms a witness for $\mathsf{b}_\kappa^{b,h}(\in^*) \leq \kappa$.

(iii) Let $\mathcal{F} \subseteq \prod b$ with $|\mathcal{F}| = \kappa$ and enumerate $\mathcal{F}$ as $\langle f_\eta \mid \eta \in \kappa\rangle$. Let C be a club set containing no successor ordinals such that h is discontinuous on C, and let $\langle \alpha_\xi \mid \xi \in \kappa\rangle$ be the increasing enumeration of $C \cup \{0\}$. For each $\xi \in \kappa$ let $\lambda_\xi = \bigcup_{\alpha \in \alpha_\xi} h(\alpha)$, where we have the convention that $\bigcup ? = 0$, then $\lambda_\xi < h(\alpha_\xi)$ for all $\xi \in \kappa$ by discontinuity. Given $\alpha \in \kappa$, let ξ be such that $\alpha \in [\alpha_\xi, \alpha_{\xi+1})$, which exists by C being club, and let $\varphi(\alpha) = \{f_\eta(\alpha) \mid \eta \in \lambda_\xi\}$. Since h is increasing, $|\varphi(\alpha)| \leq \lambda_\xi < h(\alpha_\xi) \leq h(\alpha)$, thus $\varphi \in \mathrm{Loc}_\kappa^{b,h}$. Finally h is cofinal, thus $\langle \lambda_\xi \mid \xi \in \kappa\rangle$ is cofinal, hence for every $\eta \in \kappa$ we have $f_\eta \in^* \varphi$, showing that $\kappa < \mathsf{b}_\kappa^{b,h}(\in^*)$. □

We will once again see that $\mathsf{b}_\kappa^{b,h}(\in^*)$ is nontrivial in case (iii), but first let us consider $\mathsf{d}_\kappa^{b,h}(\in^*)$.

In case (i) we have $\mathsf{d}_\kappa^{b,h}(\in^*) = 2^\kappa$. The proof is a generalisation based on Lemmas 1.8, 1.10 and 1.11 from [GS93], where the analogous theorem is proved for ${}^\omega\omega$.

Theorem 3.4.6 — *cf. [GS93, Lemmas 1.8, 1.10 and 1.11] for* ${}^\omega\omega$
If $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$ is cofinal in κ for some $\lambda < \kappa$, then $\mathsf{d}_\kappa^{b,h}(\in^*) = 2^\kappa$. ◁

Proof. We are only interested in finding a lower bound of $\mathsf{d}_\kappa^{b,h}(\in^*)$, thus by Lemma 3.3.2 we could restrict our attention to a cofinal subset of κ. We therefore assume without loss of generality that $D_\lambda = \kappa$, that is, $h(\alpha) = \lambda$ for all $\alpha \in \kappa$. To prove this lemma we assume furthermore without loss of generality that $b = h$. This suffices to prove the lemma, by Lemma 3.3.3.

Let b' be defined by $b'(\alpha) = 2^{|\alpha|}$ for all $\alpha \in \kappa$. We start with proving that $\mathsf{d}_\kappa^{b',h}(\in^*) = 2^\kappa$.

Let $\pi_\alpha : {}^\alpha 2 \rightarrowtail b'(\alpha)$ be an injection for every $\alpha \in \kappa$. For some arbitrary $g \in {}^\kappa 2$, define $f_g \in \prod b'$ to be such that $f_g(\alpha) = \pi_\alpha(g \quad \alpha)$ for all $\alpha \in \kappa$. If $g, g' \in {}^\kappa 2$ are distinct, then $g \quad \alpha \neq g' \quad \alpha$ for almost all $\alpha \in \kappa$, hence $f_g(\alpha) \neq f_{g'}(\alpha)$ for almost all $\alpha \in \kappa$. Therefore, if $\varphi \in \mathrm{Loc}_\kappa^{b',h}$, then there are at most λ many functions $g \in {}^\kappa 2$ such that $f_g \in^* \varphi$. Since ${}^\kappa 2$ cannot be the union of less than 2^κ sets of size λ, we see that $\mathsf{d}_\kappa^{b',h}(\in^*) = 2^\kappa$.

We will construct a Tukey connection $\mathbf{L}_{b',h} \preceq \mathbf{L}_{b,h}$. Let $\langle W_\alpha \mid \alpha \in \kappa\rangle$ be a partition of κ such that $|W_\alpha| = \lambda^{2^{|\alpha|}}$ and let $\left\langle \Phi^\alpha_\xi \mid \xi \in W_\alpha \right\rangle$ be an enumeration of all functions $2^{|\alpha|} \to \lambda$. We let $\rho_- : \prod b' \to \prod b$ send a function f' to the function f defined as follows: for $\xi \in \kappa$, let α be such that $\xi \in W_\alpha$, then we let $f(\xi) = \Phi^\alpha_\xi(f'(\alpha))$. We let $\rho_+ : \mathrm{Loc}^{b,h}_\kappa \to \mathrm{Loc}^{b',h}_\kappa$ send a slalom φ to the slalom φ', where $\varphi'(\alpha) = \left\{\eta \in b'(\alpha) \mid \forall \xi \in W_\alpha(\Phi^\alpha_\xi(\eta) \in \varphi(\xi))\right\}$. We show that φ' is indeed a (b', h)-slalom by proving that $|\varphi'(\alpha)| < h(\alpha) = \lambda$.

Assume towards contradiction that there exists a sequence $\langle \eta_\beta \mid \beta \in \lambda\rangle$ of distinct elements of $\varphi'(\alpha)$. We define a function $\Phi : b'(\alpha) \to \lambda$ by sending $\eta_\beta \mapsto \beta$ and $\eta \mapsto 0$ if $\eta \neq \eta_\beta$ for all $\beta \in \lambda$. Note that $\mathrm{dom}(\Phi) = b'(\alpha) = 2^{|\alpha|}$, hence there is $\xi \in W_\alpha$ such that $\Phi = \Phi^\alpha_\xi$. Since $|\varphi(\xi)| < h(\xi) = \lambda$ there is some $\beta \in \lambda \setminus \varphi(\xi)$, but then $\Phi(\eta_\beta) = \Phi^\alpha_\xi(\eta_\beta) \notin \varphi(\xi)$, which implies the contradictory $\eta_\beta \notin \varphi'(\alpha)$.

Finally we have to prove that ρ_-, ρ_+ form a Tukey connection. If we assume that $f \in^* \varphi$, then $\Phi^\alpha_\xi(f'(\alpha)) \in \varphi(\xi)$ for almost all $\xi \in \kappa$, where α is such that $\xi \in W_\alpha$. This means that for almost all $\alpha \in \kappa$ we have $\Phi^\alpha_\xi(f'(\alpha)) \in \varphi(\xi)$ for all $\xi \in W_\alpha$. Therefore, for almost all $\alpha \in \kappa$ we have $f'(\alpha) \in \varphi'(\alpha)$, showing that $f' \in^* \varphi'$. □

In case (iii) we will prove that each of $\kappa^+ < \mathfrak{b}^{b,h}_\kappa(\in^*)$ and $\mathfrak{d}^{b,h}_\kappa(\in^*) < 2^\kappa$ are consistent using a generalisation of localisation forcing in Theorem 4.3.31. This mirrors what happens for the dominating and unbounded numbers in Theorem 4.3.19.

Finally we will conclude this section with case (ii), which appears to be more complicated. Currently our best lower bound is given by the maximal size of almost disjoint families of functions.

On Almost Disjoint Families in Bounded Spaces

A family $\mathcal{A} \subseteq \prod b$ is called *almost disjoint* if $f =^\infty g$ implies that $f = g$ for all $f, g \in \mathcal{A}$.

Theorem 3.4.7

If $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$ is bounded for all $\lambda \in \kappa$ and h is increasing and continuous on a stationary set and $\mathcal{A} \subseteq \prod b$ is an almost disjoint family, then $|\mathcal{A}| \leq \mathfrak{d}^{b,h}_\kappa(\in^*)$. ◁

Proof. Since we assume $h \leq b$ and h is cofinal, b is also cofinal. Note that there exists an almost disjoint family $\mathcal{A} \subseteq \prod b$ with $|\mathcal{A}| = \kappa$: let $f_\eta \in \prod b$ send $\alpha \mapsto \eta$ if $\eta \in b(\alpha)$ and $\alpha \mapsto 0$ otherwise, then $\mathcal{A} = \{f_\eta \mid \eta \in \kappa\}$ suffices. We will therefore assume without loss of generality that $\mathcal{A}$ is an almost disjoint family with $|\mathcal{A}| \geq \kappa$.

Given $\varphi \in \mathrm{Loc}^{b,h}_\kappa$, let λ_φ be minimal such that there is stationary S_φ with $|\varphi(\alpha)| = \lambda_\varphi$ for all $\alpha \in S_\varphi$. Fix some arbitrary $A \subseteq \mathcal{A}$ with $|A| = \lambda^+_\varphi$, enumerate A as $\left\langle f_\alpha \mid \alpha \in \lambda^+_\varphi \right\rangle$ and define $\xi_{\alpha,\beta} = \min \{\xi \in \kappa \mid \forall \eta \in [\xi, \kappa)(f_\alpha(\eta) \neq f_\beta(\eta))\}$. Since $\xi_{\alpha,\beta} < \kappa$ for all distinct $\alpha, \beta \in \lambda^+_\varphi$, and $\lambda^+_\varphi < \kappa$ and κ is inaccessible, we see that $\xi = \bigcup_{\alpha\in\lambda^+_\varphi} \bigcup_{\beta\in\alpha} \xi_{\alpha,\beta} \in \kappa$. If $\eta \geq \xi$, then $f_\alpha(\eta)$ is distinct for each $\alpha \in \lambda^+_\varphi$, thus $\left|\left\{f_\alpha(\eta) \mid \alpha \in \lambda^+_\varphi\right\}\right| = \lambda^+_\varphi$. For every $\eta \in S_\varphi \setminus \xi$ there is $\alpha \in \lambda^+_\varphi$ such that $f_\alpha(\eta) \notin \varphi(\eta)$, hence by the pigeonhole principle there exists $\alpha \in \lambda^+_\varphi$ such that $f_\alpha \in^* \varphi$.

Thus, we see that $\mathcal{A}_\varphi = \{f \in \mathcal{A} \mid f \in^* \varphi\}$ has $|\mathcal{A}_\varphi| \leq \lambda_\varphi$. If $\Phi \subseteq \mathrm{Loc}^{b,h}_\kappa$ is of minimal cardinality to witnesses $\mathsf{d}^{b,h}_\kappa(\in^*)$, then $\bigcup_{\varphi\in\Phi} \mathcal{A}_\varphi = \mathcal{A}$, and thus $\kappa \leq |\mathcal{A}| \leq |\Phi| \cdot \sup_{\varphi\in\Phi} \lambda_\varphi = |\Phi|$. For the last equality, note that $\sup_{\varphi\in\Phi} \lambda_\varphi \leq \kappa$ and that this inequality would be strict if $|\Phi| < \kappa$. $\square$

Note that an almost disjoint family $\mathcal{A} \subseteq \prod b$ with $|\mathcal{A}| = 2^\kappa$ exists when there exists a continuous strictly increasing sequence $\langle \alpha_\xi \mid \xi \in \kappa\rangle$ such that $2^{|\xi|} \leq |b(\alpha_\xi)|$ for all $\xi \in \kappa$. The construction for such a family is done by fixing injections $\pi_\xi : {}^\xi 2 \rightarrowtail b(\alpha_\xi)$ and for any $f \in {}^\kappa 2$ considering the function $f' \in \prod b$ given by $f' : \alpha \mapsto \pi_\xi(f \upharpoonright \xi)$ for each $\alpha \in [\alpha_\xi, \alpha_{\xi+1})$, then $\{f' \mid f \in {}^\kappa 2\}$ forms an almost disjoint family.

Even if b does not grow fast enough such that the above construction can be done, it is still possible to add an almost disjoint family of size 2^κ with forcing. Let us focus on the case where $b = \mathsf{id} : \alpha \mapsto \alpha$ is the identity function, and look at a forcing notion $\mathsf{AD}^\lambda_\kappa$ that adds a λ-sized almost disjoint family of regressive functions, that is, elements of $\prod \mathsf{id}$.

Definition 3.4.8

The forcing notion $\mathsf{AD}^\lambda_\kappa$ has the conditions $p : X_p \times \beta_p \to \kappa$ such that $X_p \in [\lambda]^{<\kappa}$, $|X_p| \leq \beta_p \in \kappa$, and $p(\xi, \alpha) < \alpha$ for all $\xi \in X_p$ and $0 < \alpha \in \beta_p$. Given $\xi \in X_p$, we write $p_\xi : \beta_p \to \kappa$ for the (regressive) function $p_\xi : \alpha \mapsto p(\xi, \alpha)$. The ordering on $\mathsf{AD}^\lambda_\kappa$ is given by $q \leq p$ iff $p \subseteq q$ (implicitly $X_p \subseteq X_q$ and $\beta_p \leq \beta_q$), and $q_\xi(\alpha) \neq q_{\xi'}(\alpha)$ for any $\alpha \in \beta_q \setminus \beta_p$ and distinct $\xi, \xi' \in X_p$. ◁

Lemma 3.4.9

The set of all $p \in \mathsf{AD}^\lambda_\kappa$ such that $(\xi, \alpha) \in \mathrm{dom}(p)$, is dense for any $\xi \in \lambda$ and $\alpha \in \kappa$. ◁

Proof. Let $p \in \mathsf{AD}^\lambda_\kappa$ and $\beta_p \leq \alpha \in \kappa$, then we will first find $q \leq p$ with $X_q = X_p$ and $\beta_q = \alpha + 1$. Fix an enumeration $\langle \xi_\eta \mid \eta < |X_p|\rangle$ of X_p, then for any $\gamma \in [\beta_p, \alpha]$ note that $|X_p| \leq \beta_p \leq \gamma$, hence we can define $q_{\xi_\eta}(\gamma) = \eta < \gamma$, then $q \leq p$.

Next we show how to increase X_p. Let $p \in \mathsf{AD}^\lambda_\kappa$ and $\mu < \kappa$, then by the above there exists $q \leq p$ with $X_q = X_p$ and $|X_p| + \mu \leq \beta_q$. If $X \in [\lambda \setminus X_p]^\mu$, we can find $r \leq q$ with $X_r = X_p \cup X$ and $\beta_r = \beta_q$ simply by letting $r_\xi(\alpha) = 0$ for all $\alpha \in \beta_q$ and $\xi \in X$. $\square$

Lemma 3.4.10

If G is an $\mathsf{AD}^\lambda_\kappa$-generic filter over $\mathbf{V}$, and we define $f_\xi = \bigcup\{p_\xi \mid p \in G \wedge \xi \in X_p)\}$, then $f_\xi \in {}^\kappa\kappa$ is a regressive function and $\left\{f_\xi \mid \xi \in (\lambda)^{\mathbf{V}}\right\}$ is almost disjoint. ◁

Proof. It is clear from the definition of $\mathsf{AD}^\lambda_\kappa$ and the above lemma that $f_\xi \in {}^\kappa\kappa$ and that f_ξ is regressive. If $\xi, \xi' \in (\lambda)^{\mathbf{V}}$ are distinct, then let $p \in G$ be such that $\xi, \xi' \in X_p$ and let $\alpha \geq \beta_p$. For any $q \leq p$ with $q \in G$ and $\alpha \in \beta_q$ we have $q_\xi(\alpha) \neq q_{\xi'}(\alpha)$. By the above lemma there exist such $q \leq p$ with $q \in G$, hence $f_\xi(\alpha) \neq f_{\xi'}(\alpha)$ for any $\alpha > \beta_p$. Therefore f_ξ and $f_{\xi'}$ are almost disjoint. $\square$

Lemma 3.4.11

$\mathsf{AD}^\lambda_\kappa$ is $<\kappa$-closed and has the $<\kappa^+$-c.c.[3] ◁

[3] See Definitions 4.1.1 and 4.1.6 for the definitions of $<\kappa$-closure and $<\kappa^+$-c.c.

Proof. First, let us prove $<\kappa$-closure. Let $\gamma \in \kappa$ and let $\langle p^\eta \in \mathsf{AD}^\lambda_\kappa \mid \eta < \gamma \rangle$ be a descending chain of conditions. It is clear that $X = \bigcup_{\eta\in\gamma} X_{p^\eta} \in [\lambda]^{<\kappa}$ and $|X| \leq \beta = \bigcup_{\eta\in\gamma} \beta_{p^\eta} \in \kappa$, and that $p = \bigcup_{\eta\in\gamma} p^\eta : X \times \beta \to \kappa$ and $p(\xi, \eta) < \eta$ for all $(\xi, \eta) \in X \times \beta$. If $\xi, \xi' \in X_{p^\eta}$ are distinct and $\alpha \in \beta \setminus \beta_{p^\eta}$, then there is $\zeta > \eta$ such that $\alpha \in \beta_{p^\zeta}$. Since the chain of conditions is descending we have $p^\zeta \leq p^\eta$, which implies that $p_\xi(\alpha) = p^\zeta_\xi(\alpha) \neq p^\zeta_{\xi'}(\alpha) = p_{\xi'}(\alpha)$, so $p \leq p_\eta$.

As for $<\kappa^+$-c.c., suppose $\mathcal{B} \subseteq \mathsf{AD}^\lambda_\kappa$ is a subset with $|\mathcal{B}| = \kappa^+$. We let $\mathcal{A} = \{\operatorname{dom}(p) \mid p \in \mathcal{B}\}$, then by $|X_p| < \kappa$ and $\beta_p < \kappa$ for each $p \in \mathcal{B}$ we see that $\mathcal{A}$ is a family of sets of cardinality $<\kappa$. Applying the Δ-system lemma (see Lemma 4.1.16) on $\mathcal{A}$, we know that there exists $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $|\mathcal{B}_0| = \kappa^+$ and some sets $X \in [\lambda]^{<\kappa}$ and $\beta \in \kappa$ such that $\operatorname{dom}(p) \cap \operatorname{dom}(p') = X \times \beta$ for all distinct $p, p' \in \mathcal{B}_0$.

If $X = ?$, then any $p, p' \in \mathcal{B}_0$ have some $q \leq p$ and $q' \leq p'$ such that $|X_p| + |X_{p'}| \leq \beta_q = \beta_{q'}$ using Lemma 3.4.9. Then $q \cup q' \leq p$ and $q \cup q' \leq p'$, thus $\mathcal{B}$ is not an antichain.

On the other hand, if $X \neq ?$, then $\beta = \beta_p = \beta_{p'}$ for all $p, p' \in \mathcal{B}_0$. Since $|X \times \beta| < \kappa$ and $\kappa^{<\kappa} = \kappa$, there exists $\mathcal{B}_1 \subseteq \mathcal{B}_0$ with $|\mathcal{B}_1| = \kappa^+$ such that $p \;\; (X \times \beta) = p' \;\; (X \times \beta)$ for all $p, p' \in \mathcal{B}_1$. Once again, using Lemma 3.4.9 we can see that p, p' are compatible, and thus $\mathcal{B}$ is not an antichain. □

Corollary 3.4.12

$\mathsf{AD}^\lambda_\kappa$ preserves cardinals and cofinality and does not add any elements to ${}^{<\kappa}\kappa$. ◁

Theorem 3.4.13

It is consistent that there exists an almost disjoint family $\mathcal{A} \subseteq \prod \mathsf{id}$ of cardinality $2^\kappa = \lambda$ for any λ with $\operatorname{cf}(\lambda) > \kappa$. ◁

Proof. Starting with a model where $\mathbf{V}$ "$2^\kappa \leq \lambda$", let G be $\mathsf{AD}^\lambda_\kappa$-generic over $\mathbf{V}$, and let $\mathcal{F} = \left\{ f_\xi \mid \xi \in (2^\kappa)^{\mathbf{V}} \right\}$ be the almost disjoint family described in Lemma 3.4.10. An argument by counting names shows that $(2^\kappa)^{\mathbf{V}} \leq (2^\kappa)^{\mathbf{V}[G]} = \lambda$. □

Although the $<\kappa$-closure implies that $\mathsf{AD}^\lambda_\kappa$ preserves the inaccessibility of κ, the same cannot be said for stronger large cardinal assumptions on κ. Indeed, it is inconsistent that an almost disjoint family $\mathcal{A} \subseteq \prod \mathsf{id}$ of size larger than κ exists for measurable cardinals, as was pointed out to me by Jing Zhang.

Theorem 3.4.14

If κ is measurable, $b \in {}^\kappa\kappa$ is continuous and $\mathcal{A} \subseteq \prod b$ is almost disjoint, then $|\mathcal{A}| \leq \kappa$. ◁

Proof. Let $\mathcal{U} \subseteq \mathcal{P}(\kappa)$ be a $<\kappa$-complete nonprincipal normal ultrafilter on κ. Assume towards contradiction that $\mathcal{A} \subseteq \prod b$ is an almost disjoint family with $|\mathcal{A}| = \kappa^+$. Since b is continuous, the set of fixed points of b contains a club set, thus every $f \in \mathcal{A}$ is regressive on a club set. Therefore there exists $X_f \in \mathcal{U}$ such that f is regressive on X_f, and since $\mathcal{U}$ is normal there exists $Y_f \in \mathcal{U} \;\; X_f$ and $\gamma_f \in \kappa$ such that $\operatorname{ran}(f \;\; Y_f) = \{\gamma_f\}$. By the pigeonhole principle there exists $\mathcal{A}' \subseteq \mathcal{A}$ and $\gamma \in \kappa$ with $|\mathcal{A}'| = |\mathcal{A}|$ such that $\gamma_f = \gamma$ for all $f \in \mathcal{A}'$. Then for any distinct $f, f' \in \mathcal{A}'$ we have $Y_f \cap Y_{f'} \in \mathcal{U}$, contradicting that f and f' are almost disjoint. □

In the end, we do not know whether the conditions of (ii) of Theorem 3.4.5 imply that $\mathsf{d}_\kappa^{b,h}(\in^*)$ is trivial.

Antilocalisation & Antiavoidance

We can give similar results for $\mathsf{d}_\kappa^{b,h}(\ni^\infty)$ and $\mathsf{b}_\kappa^{b,h}(\ni^\infty)$ to what we discussed in the previous sections. Firstly, due to Lemmas 3.3.2, 3.3.3 and 3.4.15 below, we will once again assume that $h \leq b$ is always the case.

Lemma 3.4.15
$b <^\infty h$ if and only if $\mathsf{d}_\kappa^{b,h}(\ni^\infty) = 1$.
If $b <^\infty h$, then $\mathsf{b}_\kappa^{b,h}(\ni^\infty)$ is undefined.

A comprehensive overview of the trivial values for these cardinals in the classical context has been given by Cardona & Mejía in [CM19, Section 3]. However, the classical characterisation uses a substantial amount of finite arithmetic and thus appears quite different from the characterisation of the trivial values of $\mathsf{b}_\kappa^{b,h}(\ni^\infty)$, given below. We once again have three cases, making this similar to Theorems 3.4.2 and 3.4.5.

Theorem 3.4.16
For each $\lambda < \kappa$ define $D_\lambda = \{\alpha \in \kappa \mid b(\alpha) \leq \lambda\} \cup \{\alpha \in \kappa \mid h(\alpha) = b(\alpha) \text{ and } \mathrm{cf}(b(\alpha)) \leq \lambda\}$.

(i) If there exists a least cardinal $\lambda < \kappa$ such that D_λ is cofinal in κ, then $\mathsf{b}_\kappa^{b,h}(\ni^\infty) = \lambda$.
(ii) If D_λ is bounded for all $\lambda < \kappa$ and there is a stationary set S such that
   (a) b is continuous on S, *or*
   (b) for each $\xi \in S$ there exists $\alpha_\xi \geq \xi$ with $h(\alpha_\xi) = b(\alpha_\xi)$ and $\mathrm{cf}(b(\alpha_\xi)) \leq \xi$,
   then $\mathsf{b}_\kappa^{b,h}(\ni^\infty) = \kappa$.
(iii) If D_λ is bounded for all $\lambda < \kappa$ and there is a club set C such that
   (a) b is discontinuous on C, *and*
   (b) for each $\xi \in C$ and $\alpha \geq \xi$, if $h(\alpha) = b(\alpha)$, then $\mathrm{cf}(b(\alpha)) > \xi$,
   then $\kappa^+ \leq \mathsf{b}_\kappa^{b,h}(\ni^\infty)$. ◁

Proof. (i) Assume that λ is minimal such that D_λ is cofinal. Let $D'_\lambda = \{\alpha \in \kappa \mid b(\alpha) = \lambda\}$ and $D''_\lambda = \{\alpha \in \kappa \mid h(\alpha) = b(\alpha) \text{ and } \mathrm{cf}(b(\alpha)) = \lambda\}$. Note that at least one of D'_λ or D''_λ is cofinal. We will assume without loss of generality that $D_\lambda = D'_\lambda \cup D''_\lambda$.

First, we will show that $\mathsf{b}_\kappa^{b,h}(\ni^\infty) \leq \lambda$.

If $\alpha \in D''_\lambda$, then $\mathrm{cf}(b(\alpha)) = \lambda$, thus we can find a continuous sequence $\langle \beta_\xi^\alpha \mid \xi < \lambda \rangle$ that is cofinal in $b(\alpha)$. In the other case where $\alpha \in D'_\lambda$, let $\beta_\xi^\alpha = \xi$ for all $\xi < \lambda = b(\alpha)$.

Since $|[\beta_\xi^\alpha, \beta_{\xi+1}^\alpha)| < h(\alpha) \leq b(\alpha)$ for all $\alpha \in D_\lambda$, we can pick some $\varphi_\xi \in \mathrm{Loc}_\kappa^{b,h}$ for each $\xi < \lambda$ such that $\varphi_\xi(\alpha) = [\beta_\xi^\alpha, \beta_{\xi+1}^\alpha)$ for all $\alpha \in D_\lambda$. For $f \in \prod b$, let $F_f : D_\lambda \to \lambda$ map $\alpha \in D_\lambda$ to the $\xi \in \lambda$ such that $f(\alpha) \in \varphi_\xi(\alpha)$, then by the pigeonhole principle $F_f^{-1}(\xi)$ is cofinal for some $\xi \in \lambda$, and thus for this ξ we have $f \in^\infty \varphi_\xi$. Therefore $\{\varphi_\xi \mid \xi < \lambda\}$ witnesses that $\mathsf{b}_\kappa^{b,h}(\ni^\infty) \leq \lambda$.

Next we show that $\lambda \leq \mathfrak{b}_\kappa^{b,h}(\ni^\infty)$. Let $\{\varphi_\xi \mid \xi \in \mu\} \subseteq \mathrm{Loc}_\kappa^{b,h}$ for some $\mu < \lambda$.

If D'_λ is cofinal, then we must have $b(\alpha) \geq \lambda$ for almost all $\alpha \in \kappa$ by minimality of λ. Suppose that $\lambda \leq b(\alpha)$ and $\bigcup_{\xi<\mu} \varphi_\xi(\alpha) = b(\alpha)$, then $\mathrm{cf}(b(\alpha)) \leq \mu$, and furthermore for every $\nu < b(\alpha)$ there is some $\xi \in \mu$ such that $\nu \leq |\varphi_\xi(\alpha)|$, implying that $h(\alpha) = b(\alpha)$. But then $\alpha \in D''_\mu$, which is bounded by minimality of λ. Therefore $\bigcup_{\xi<\mu} \varphi(\alpha) \neq b(\alpha)$ for almost all $\alpha \in \kappa$.

If D'_λ is bounded and D''_λ is cofinal, then for almost all $\alpha \notin D''_\lambda$ we must have $h(\alpha) \neq b(\alpha)$ or $\mathrm{cf}(b(\alpha)) > \lambda$. Hence we have $\max\{h(\alpha), \lambda\} < b(\alpha)$ or $\mathrm{cf}(b(\alpha)) \geq \lambda$ for almost all $\alpha \in \kappa$. If $\max\{h(\alpha), \lambda\} < b(\alpha)$, then also $\mu \cdot h(\alpha) < b(\alpha)$, thus $\bigcup_{\xi<\mu} \varphi_\xi(\alpha) \neq b(\alpha)$. On the other hand if $\mathrm{cf}(b(\alpha)) \geq \lambda$, then clearly $\bigcup_{\xi<\mu} \varphi_\xi(\alpha) \neq b(\alpha)$ as well.

In either case we find $f \in \prod b$ such that $f \in^\infty \varphi_\xi$ for all $\xi < \mu$, and consequently $\mu < \mathfrak{b}_\kappa^{b,h}(\ni^\infty)$.

(ii) We first prove that $\mathfrak{b}_\kappa^{b,h}(\ni^\infty) \leq \kappa$ in case (a), then in case (b), and then we show the reverse direction, that $\kappa \leq \mathfrak{b}_\kappa^{b,h}(\ni^\infty)$.

(a) Assume b is increasing and continuous on stationary $S \subseteq \kappa$. Then the set of fixed points $S' = \{\alpha \in S \mid b(\alpha) = \alpha\}$ is also stationary.

For each $\eta \in \kappa$ let $\varphi_\eta \in \mathrm{Loc}_\kappa^{b,h}$ be defined as $\varphi_\eta(\xi) = \{\eta\}$ if $\eta < b(\xi)$ and arbitrary otherwise. If $f \in \prod b$, then $f(\xi) < b(\xi)$ for every $\xi \in \kappa$, therefore f is regressive on S'. By Fodor's lemma then there exists stationary $S'' \subseteq S'$ such that $f \quad S''$ is constant. Let η be such that $f(\xi) = \eta$ for all $\eta \in S''$, then clearly $f \in^\infty \varphi_\eta$. Hence $\{\varphi_\eta \mid \eta < \kappa\}$ witnesses $\mathfrak{b}_\kappa^{b,h}(\ni^\infty) \leq \kappa$.

(b) Assume S is stationary and for each $\xi \in S$ let $\alpha_\xi \geq \xi$ be such that $h(\alpha_\xi) = b(\alpha_\xi)$ and $\mathrm{cf}(b(\alpha_\xi)) \leq \xi$. We define for each $\eta \in \kappa$ a slalom $\varphi_\eta \in \mathrm{Loc}_\kappa^{b,h}$ in such a way that for each $\xi \in S$ we have $\bigcup_{\eta<\xi} \varphi_\eta(\alpha_\xi) = b(\alpha_\xi)$, which is possible since $\mathrm{cf}(b(\alpha_\xi)) \leq \xi$ and $h(\alpha_\xi) = b(\alpha_\xi)$.

If $f \in \prod b$, then for each $\xi \in S$ we define $f'(\xi)$ to be the least $\eta < \xi$ such that $f(\alpha_\xi) \in \varphi_\eta(\alpha_\xi)$. Now f' is regressive on S, so by Fodor's lemma there exists stationary $S' \subseteq S$ such that $f' \quad S'$ is constant, say with value η. Then $f(\alpha_\xi) \in \varphi_\eta(\alpha_\xi)$ for each $\xi \in S'$, thus $f \in^\infty \varphi_\eta$. Hence $\{\varphi_\eta \mid \eta < \kappa\}$ witnesses $\mathfrak{b}_\kappa^{b,h}(\ni^\infty) \leq \kappa$.

Finally for the reverse direction, let $\lambda < \kappa$ and $\{\varphi_\xi \mid \xi < \lambda\} \subseteq \mathrm{Loc}_\kappa^{b,h}$. Since D_λ is bounded, we can find $\alpha_0 \in \kappa$ such that for every $\alpha > \alpha_0$ we have $\lambda < b(\alpha)$ and either $h(\alpha) < b(\alpha)$ or $\lambda < \mathrm{cf}(b(\alpha))$. If $h(\alpha) < b(\alpha)$, then $\lambda \cdot h(\alpha) < b(\alpha)$, meaning $\bigcup_{\xi<\lambda} \varphi_\xi(\alpha) \neq b(\alpha)$. On the other hand, if $\lambda < \mathrm{cf}(b(\alpha))$, then by $|\varphi_\xi(\alpha)| < h(\alpha) \leq b(\alpha)$ we see that once again $\bigcup_{\xi<\lambda} \varphi_\xi(\alpha) \neq b(\alpha)$. Hence we can construct $f \in \prod b$ such that $f \in^\infty \varphi_\xi$ for all $\xi < \lambda$, which proves that $\kappa \leq \mathfrak{b}_\kappa^{b,h}(\ni^\infty)$.

(iii) Let C be a club set with the properties mentioned in (iii) and let $\{\varphi_\eta \mid \eta \in \kappa\} \subseteq \mathrm{Loc}_\kappa^{b,h}$. We can enumerate C increasingly as $\langle \alpha_\xi \mid \xi \in \kappa \rangle$, then every $\alpha \in \kappa$ has some $\xi \in \kappa$ such that $\alpha \in [\alpha_\xi, \alpha_{\xi+1})$. Note that $|\bigcup_{\eta\in\xi} \varphi_\eta(\alpha)| \leq |\xi| \cdot \sup_{\eta\in\xi} |\varphi_\eta(\alpha)|$. Since b is increasing and discontinuous on C and $\xi \leq \alpha_\xi \leq \alpha$, we see that $\xi < b(\alpha_\xi) \leq b(\alpha)$. If $h(\alpha) < b(\alpha)$, then it is clear that $\sup_{\eta\in\xi} |\varphi_\eta(\alpha)| < b(\alpha)$ as well. Else, if $h(\alpha) = b(\alpha)$, then by the properties of C we see that $\mathrm{cf}(b(\alpha)) > \alpha_\xi \geq \xi$, thus it also follows that $\sup_{\eta\in\xi} |\varphi_\eta(\alpha)| < b(\alpha)$. We can therefore conclude that $|\bigcup_{\eta\in\xi} \varphi_\eta(\alpha)| < b(\alpha)$ for all $\alpha \in \kappa$, and thus we can define $f(\alpha)$ to be some value in $b(\alpha)$ that lies outside of $\varphi_\eta(\alpha)$ for each $\eta < \xi$. Then clearly $f \in^\infty \varphi_\eta$ for all $\eta \in \kappa$. □

It is not known whether cases (i) and (ii) of Theorem 3.4.16 imply that $\mathfrak{d}^{b,h}_\kappa(\ni^\infty) = 2^\kappa$. If we consider the constant function $\overline{2}$, then we see that $\mathfrak{d}^{b,\overline{2}}_\kappa(\ni^\infty) \leq \mathfrak{d}^{b,h}_\kappa(\ni^\infty)$ by Lemma 3.3.3. Since $\mathfrak{d}^{b,\overline{2}}_\kappa(\ni^\infty) = \mathfrak{d}^{b}_\kappa(=^\infty)$ (by Lemma 3.3.4), we will discuss the cases (i) and (ii) in the next section. Let us mention here, that if, by exception, we allow that $b(\alpha)$ is finite for cofinally many α, then we may use a connection with localisation cardinals to prove that $\mathfrak{d}^{b,h}_\kappa(\ni^\infty) = 2^\kappa$ (see Lemma 3.4.19). We will also give a partial answer to case (i) with the following lemma:

Lemma 3.4.17
If λ is regular and $D = \{\alpha \in \kappa \mid b(\alpha) = h(\alpha) = \lambda\}$ is cofinal, then $\mathbf{D}^{\overline{\lambda}}_\kappa \preceq \mathbf{AL}^{b,h}_\kappa$. ◁

Proof. Enumerate D as $\{\alpha_\xi \mid \xi \in \kappa\}$. We need $\rho_- : {}^\kappa\lambda \to \mathrm{Loc}^{b,h}_\kappa$ and $\rho_+ : \prod b \to {}^\kappa\lambda$.

Given $g \in {}^\kappa\lambda$, let $\rho_-(g) \in \mathrm{Loc}^{b,h}_\kappa$ be such that $\rho_-(g) : \alpha_\xi \mapsto g(\xi)$ for all $\xi \in \kappa$. Given $f \in \prod b$, let $\rho_+(f) : \xi \mapsto f(\alpha_\xi)$ for each $\xi \in \kappa$. Since $\alpha_\xi \in D$ for each $\xi \in \kappa$, we have $h(\alpha_\xi) = b(\alpha_\xi) = \lambda$, thus ρ_- and ρ_+ are well-defined. If $\varphi = \rho_-(g)$ and $f' = \rho_+(f)$, and $f \in^\infty \varphi$, then it is easy to see that $g \leq^* f'$, so this is a Tukey connection. □

Since we already know that $\mathfrak{d}^{\overline{\lambda}}_\kappa(\leq^*) = 2^\kappa$ from Theorem 3.4.3, the assumptions of the above lemma imply that $\mathfrak{d}^{b,h}_\kappa(\ni^\infty) = 2^\kappa$.

Eventual Difference & Cofinal Equality

Since antilocalisation is a special case of eventual difference, it follows that Theorem 3.4.16 immediately gives us the following characterisation of the trivial values of $\mathfrak{b}^b_\kappa(=^\infty)$, since we can ignore the case where $h(\alpha) = b(\alpha)$ if we let $h = \overline{2}$.

Corollary 3.4.18 — *cf. Theorem 3.4.16*
For each $\lambda < \kappa$ define $D_\lambda = \{\alpha \in \kappa \mid b(\alpha) \leq \lambda\}$.

(i) If there exists a least cardinal $\lambda < \kappa$ such that D_λ is cofinal in κ, then $\mathfrak{b}^b_\kappa(=^\infty) = \lambda$.
(ii) If D_λ is bounded for all $\lambda < \kappa$ and b is continuous on a stationary set, then $\mathfrak{b}^b_\kappa(=^\infty) = \kappa$.
(iii) If D_λ is bounded for all $\lambda < \kappa$ and b is discontinuous on a club set, then $\kappa^+ \leq \mathfrak{b}^b_\kappa(=^\infty)$.

We can use Theorem 3.4.6 and a special connection between localisation and antilocalisation for functions with finite values to prove the following lemma, giving a condition for which $\mathfrak{d}^b_\kappa(=^\infty)$ and $\mathfrak{d}^{b,h}_\kappa(\ni^\infty)$ are trivial. Note that this is significantly weaker than the assumption in case (i), since b is not only bounded on a cofinal set, but even finite cofinally often.

Lemma 3.4.19
If $D = \{\alpha \in \kappa \mid b(\alpha) < \omega\}$ is cofinal, then $\mathfrak{d}^b_\kappa(=^\infty) = \mathfrak{d}^{b,h}_\kappa(\ni^\infty) = 2^\kappa$. ◁

Proof. We want to give a lower bound to $\mathfrak{d}^{b,h}_\kappa(\ni^\infty)$, thus by Lemma 3.3.3 we may assume that $h = \overline{2}$, and moreover by Lemma 3.3.2 and D being cofinal, we may assume that $b = \overline{n}$ for some $n \in \omega$.

Note that $\mathbf{L}_{\overline{n},\overline{n}} \equiv \mathbf{AL}_{\overline{n},\overline{2}}$. Namely, if $f \in \prod b$, let $\varphi_f : \alpha \mapsto n \setminus \{f(\alpha)\}$, then $\varphi_f \in \mathrm{Loc}^{\overline{n},\overline{n}}_\kappa$, and on the other hand, if $\varphi \in \mathrm{Loc}^{\overline{n},\overline{2}}_\kappa$, then let f_φ be the unique function such that $\varphi(\alpha) = \{f_\varphi(\alpha)\}$ for all $\alpha \in \kappa$. It is clear that $f \in^\infty \varphi$ if and only if $f_\varphi \in^* \varphi_f$.

It follows that $2^\kappa = \mathsf{d}_\kappa^{\overline{n},\overline{n}}(\in^*) \leq \mathsf{d}_\kappa^{\overline{n}}(=^\infty) \leq \mathsf{d}_\kappa^{b}(=^\infty) = \mathsf{d}_\kappa^{b,\overline{2}}(\ni^\infty) \leq \mathsf{d}_\kappa^{b,h}(\ni^\infty)$ if D is cofinal. Here the first equality is given by Theorem 3.4.6. □

The above argument is able to relate eventual difference to localisation by removing a single element from a finite set, which changes its cardinality. Naturally, removing a single element from an infinite set does not change the cardinality of the infinite set, thus the technique from this proof does not give us insight on how to prove $\mathsf{d}_\kappa^{b}(=^\infty) = 2^\kappa$ if b is only bounded on a cofinal set by an infinite value.

It is also unclear whether $\mathsf{d}_\kappa^{b}(=^\infty) = 2^\kappa$ in case (ii) of Corollary 3.4.18.

3.5. Open Questions

Although Theorems 3.4.2, 3.4.5 and 3.4.16 give us a trichotomy on the b-side of the bounded cardinal characteristics, the d-side has only been partially solved.

For $\mathsf{d}_\kappa^{b}(\leq^*)$ the remaining case was case (ii) of Theorem 3.4.2.

Question 3.5.1
Let $D_\lambda = \{\alpha \in \kappa \mid \mathrm{cf}(b(\alpha)) = \lambda\}$ be bounded for all regular $\lambda < \kappa$ and S be stationary such that for each $\xi \in S$ there exists $\alpha_\xi \geq \xi$ with $\mathrm{cf}(b(\alpha_\xi)) \leq \xi$. Is $\mathsf{d}_\kappa^{b}(\leq^*) < 2^\kappa$ consistent? ◁

For $\mathsf{d}_\kappa^{b,h}(\in^*)$ we have a lower bound in terms of the cardinality of an almost disjoint family $\mathcal{A} \subseteq \prod b$. However, we also saw that if κ is measurable, then $|\mathcal{A}| = \kappa$ for any almost disjoint family $\mathcal{A} \subseteq \prod b$ for continuous b. Therefore, we cannot use the size of almost disjoint families to show that $\mathsf{d}_\kappa^{b,h}(\in^*) = 2^\kappa$ holds. This motivates the following question:

Question 3.5.2
Is $\mathsf{d}_\kappa^{b,h}(\in^*) < 2^\kappa$ consistent with b continuous on a club set? ◁

Finally, for $\mathsf{d}_\kappa^{b,h}(\ni^\infty)$ and $\mathsf{d}_\kappa^{b}(=^\infty)$ we do not know the answer in neither case (i) nor case (ii) of Theorem 3.4.16 and Corollary 3.4.18.

Question 3.5.3
Is $\mathsf{d}_\kappa^{b}(=^\infty) < 2^\kappa$ consistent in case (i) or (ii) of Corollary 3.4.18? ◁

4

Forcing Notions and κ-Reals

In this section we will discuss forcing notions and how they affect our cardinal characteristics. Our forcing notions are generalisations of well-known classical forcing notions, such as Cohen, Hechler or Sacks forcing, which could each be considered as arboreal forcing notions. Particularly, conditions could be represented by trees on ${}^{<\omega}\omega$ ordered by inclusion.

Obtaining a useful forcing notion in the context of ${}^{\kappa}\kappa$ is sometimes not as simple as replacing ${}^{<\omega}\omega$ with ${}^{<\kappa}\kappa$ and we may need to stipulate additional requirements to ensure the forcing notion behaves as intended. Essentially, the reason for this is that in trees on ${}^{<\kappa}\kappa$ there are nodes of limit height. We will see that concepts like ultrafilters, stationary sets and large cardinals play a role in defining forcing notions with desirable properties.

In order to prove independence results for the bounded cardinal characteristics from Chapter 3, we will introduce bounded variants of our higher forcing notions as well. The specific bound that is chosen for the forcing notion may have the effect that it increases cardinal characteristics only for a select number of parameters, allowing us to separate multiple cardinal characteristics of the same type. These kinds of results will be the topic in the last two Chapters 5 and 6.

The forcing notions and properties described in this chapter are generalisations of very well-known forcing notions and properties, and even most of the higher forcing notions have been considered by others before. Most results of this chapter could therefore be attributed to the literature or to folklore, or are very similar to analogous results on ${}^{\omega}\omega$.

Nota Bene! We will assume for the remainder of the chapter without mention that b, h are increasing cofinal cardinal functions. This also extends to indexed or accented functions using the symbols b, h, such as b_ξ, h', and so on.

4.1. Properties of Forcing Notions

Before we introduce the forcing notions, we will discuss some properties that forcing notions could possess. We will generally want our forcing notions to preserve cardinals and not to effect the sets in $\mathbf{V}_\kappa$, but also to add new κ-reals in such a way that we can prove independence results.

For the preservation of cardinals, it will be enough to show that a forcing notion is $<\kappa$-distributive and $<\kappa^+$-c.c. We introduce these two concepts in the following two subsections. In some cases we cannot show that a forcing notion is $<\kappa^+$-c.c., and we will show preservation of cardinals using certain boundedness properties, introduced in the third subsection.

Closure and Distributivity

We consider three types of closure properties of a forcing notion P, namely $<\kappa$-closure, strategic $<\kappa$-closure and $<\kappa$-distributivity.

Definition 4.1.1

A forcing notion P is *$<\kappa$-closed* if for every sequence of conditions $\langle p_\alpha \mid \alpha \in \gamma\rangle$ such that $\gamma < \kappa$ and $p_\beta \leq p_\alpha$ for each $\alpha < \beta < \gamma$ there exists a condition $p \in \mathsf{P}$ with $p \leq p_\alpha$ for all $\alpha < \gamma$. $\triangleleft$

Definition 4.1.2

We define a game $\mathcal{G}(\mathsf{P}, p)$ of length κ between Black and White. We consider $p \in \mathsf{P}$ to be the first Black move of $\mathcal{G}(\mathsf{P}, p)$ (let's say at stage -1). At stage $\alpha \in \kappa$, White chooses a condition p_α stronger than all previous Black moves and Black subsequently chooses $p'_\alpha \leq p_\alpha$. White wins $\mathcal{G}(\mathsf{P}, p)$ if White can make moves at every stage $\alpha \in \kappa$.

When White has a winning strategy for $\mathcal{G}(\mathsf{P}, p)$ for all $p \in \mathsf{P}$, we call P *strategically $<\kappa$-closed*. $\triangleleft$

Definition 4.1.3

A forcing notion P is *$<\kappa$-distributive* if for any sequence $\langle D_\alpha \mid \alpha \in \lambda\rangle$ of open dense sets with $\lambda < \kappa$, also $\bigcap_{\alpha\in\lambda} D_\alpha$ is dense. $\triangleleft$

These properties are easily seen to be progressively stronger: any $<\kappa$-closed forcing notion is strategically $<\kappa$-closed, and any strategically $<\kappa$-closed forcing notion is $<\kappa$-distributive.

The reason we are interested in these properties, is that they suffice to prove that P does not collapse any cardinals $\leq\kappa$, since we can prove that any function $\lambda \to \mathbf{V}$ from the extension is already present in the ground model for any $\lambda < \kappa$. This holds particularly for any function $\lambda \to \mu$ for all $\lambda < \mu \leq \kappa$, and thus no such function can be surjective.

Lemma 4.1.4 — *Folklore, see e.g. [Jec86, Theorem 2.10]*

A forcing notion P is $<\kappa$-distributive if and only if for all $\lambda < \kappa$ and $f \in \mathbf{V}^{\mathsf{P}}$ with $f : \lambda \to \mathbf{V}$, we have $f \in \mathbf{V}$.

Corollary 4.1.5

If P is $<\kappa$-distributive, it preserves cardinals $\leq \kappa$. $\triangleleft$

Chain Conditions, Centredness and Calibre

In order to also preserve cardinals $> \kappa$, we introduce chain conditions as well as a couple of stronger properties. By an argument of counting nice names, chain conditions imply that a forcing notion preserves cardinals.

Definition 4.1.6

A forcing notion P is *$<\lambda$-c.c.* (i.e. $<\lambda$-chain condition) if all antichains in P have size $<\lambda$. $\triangleleft$

Lemma 4.1.7 — *Folklore, see e.g. [Jec86, Part I Theorem 2.14]*

If P has the $<\lambda$-c.c. and $\mathrm{cf}(\lambda) > \kappa$, then every $f \in \mathbf{V}^{\mathsf{P}}$ with $f : \kappa \to \lambda$ is bounded.

Since bounded functions cannot be surjective, this means that the chain condition implies preservation of cardinals.

Corollary 4.1.8
If P is $<\kappa^+$-c.c., it preserves cardinals $> \kappa$. $\triangleleft$

There exist several properties that are stronger than chain conditions. We introduce two such properties.

Definition 4.1.9
A subset Q of a forcing notion P is called $<\lambda$-***linked*** if for every $P \in [Q]^{<\lambda}$ there exists $q \in \mathsf{P}$ such that $q \leq p$ for all $p \in P$.

We say that P is $(\kappa, <\lambda)$-***centred*** if $\mathsf{P} = \bigcup_{\alpha\in\kappa} P_\alpha$ such that P_α is $<\lambda$-linked for every $\alpha \in \kappa$. $\triangleleft$

Definition 4.1.10
A subset Q of a forcing notion P is called λ-***calibre*** if for every $P \in [Q]^\lambda$ there exist $P' \in [P]^\lambda$ and $q \in \mathsf{P}$ such that $q \leq p$ for all $p \in P'$.

We say that P is (κ, λ)-***calibre*** if $\mathsf{P} = \bigcup_{\alpha\in\kappa} P_\alpha$ such that P_α is λ-calibre for every $\alpha \in \kappa$. $\triangleleft$

An antichain can contain at most one condition per $<\lambda$-linked set, and has $< \lambda$ many elements per λ-calibre set, thus it follows that $(\kappa, <\lambda)$-centred or (κ, λ)-calibre forcing notions are $<\kappa^+$-c.c. for $3 \leq \lambda \leq \kappa^+$.

Boundedness

Finally we will mention three related properties that tell us that new κ-reals that are added by a forcing notion are in a certain sense bound by κ-reals from the ground model. Especially the Sacks property will have a crucial role in Chapter 5.

Definition 4.1.11
A forcing notion P is ${}^\kappa\kappa$-***bounding*** if for every name $\dot f$ and condition $p \in \mathsf{P}$ such that p “$\dot f \in {}^\kappa\kappa$” there exists some $g \in {}^\kappa\kappa$ in the ground model and $q \leq p$ such that q “$\dot f <^* g$”. $\triangleleft$

Definition 4.1.12
Let $b, h \in {}^\kappa\kappa$. A forcing notion P has the (b, h)-***Laver property*** if for every name $\dot f$ and condition $p \in \mathsf{P}$ such that p “$\dot f \in \prod b$” there exists some $\varphi \in \mathrm{Loc}_\kappa^{b,h}$ in the ground model and $q \leq p$ such that q “$\dot f \in^* \varphi$”.

We say that P has the h-***Laver property*** if P has the (b, h)-Laver property for all $b \in {}^\kappa\kappa$. $\triangleleft$

Definition 4.1.13
Let $h \in {}^\kappa\kappa$. A forcing notion P has the h-***Sacks property*** if for every name $\dot f$ and condition $p \in \mathsf{P}$ such that p “$\dot f \in {}^\kappa\kappa$” there exists some $\varphi \in \mathrm{Loc}_\kappa^h$ in the ground model and $q \leq p$ such that q “$\dot f \in^* \varphi$”. $\triangleleft$

The h-Sacks property could be seen as the combination of h-Laver properties and the ${}^\kappa\kappa$-bounding property, and indeed, these two things are equivalent.

Lemma 4.1.14 — *Folklore, cf. [BJ95, Lemma 6.3.38] for ${}^\omega\omega$*
A forcing notion has the h-Sacks property iff it is ${}^\kappa\kappa$-bounding and has the h-Laver property.

Iterations and Products

In this subsection we discuss how some of the previously defined properties behave under products or iterations with $<\kappa$- and $\leq\kappa$-support. We will first establish our notation for products and iterations.

Notation

Let $\mathcal{A}$ be a set of ordinals and P_ξ be a forcing notion for each $\xi \in \mathcal{A}$. We will denote $\leq_{\mathsf{P}_\xi}$ as $\leq^\xi$, or more commonly as $\leq$ when there is no possibility for confusion, and we denote $\mathbb{1}_{\mathsf{P}_\xi}$ as $\mathbb{1}_\xi$. Consider an element of the full product $p \in \prod_{\xi\in\mathcal{A}} \mathsf{P}_\xi$, that is, p is a function with $\mathrm{dom}(p) = \mathcal{A}$ such that $p(\xi) \in \mathsf{P}_\xi$ for each $\xi \in \mathcal{A}$. We define the ***support*** of p as $\mathrm{supp}(p) = \{\xi \in \mathcal{A} \mid p(\xi) \neq \mathbb{1}_\xi\}$. We define the $\leq\kappa$-support product as follows:[1]

$$\overline{\mathsf{P}} = \textstyle\prod^{\leq\kappa}_{\xi\in\mathcal{A}} \mathsf{P}_\xi = \left\{ p \in \prod_{\xi\in\mathcal{A}} \mathsf{P}_\xi \;\middle|\; |\mathrm{supp}(p)| \leq \kappa \right\}.$$

If $p, q \in \overline{\mathsf{P}}$, then $q \leq_{\overline{\mathsf{P}}} p$ iff $q(\xi) \leq^\xi p(\xi)$ for all $\xi \in \mathcal{A}$. We will again write $q \leq p$ instead of $q \leq_{\overline{\mathsf{P}}} p$ if the context is clear, and we will write $\overline{\mathbb{1}}$ for $\mathbb{1}_{\overline{\mathsf{P}}}$.

Suppose $X \subseteq \overline{\mathsf{P}}$ and $\mathcal{B} \subseteq \mathcal{A}$, then we write $X \ \ \mathcal{B} = \{p \ \ \mathcal{B} \mid p \in X\}$. We will write $\mathcal{B}^{\mathrm{c}} = \mathcal{A} \setminus \mathcal{B}$. Note that if $G \subseteq \overline{\mathsf{P}}$ is $\overline{\mathsf{P}}$-generic over $\mathbf{V}$, then clearly $\overline{\mathsf{P}}$ and $(\overline{\mathsf{P}} \ \ \mathcal{B}) \times (\overline{\mathsf{P}} \ \ \mathcal{B}^{\mathrm{c}})$ are forcing equivalent, $(G \ \ \mathcal{B}) \times (G \ \ \mathcal{B}^{\mathrm{c}})$ is $(\overline{\mathsf{P}} \ \ \mathcal{B}) \times (\overline{\mathsf{P}} \ \ \mathcal{B}^{\mathrm{c}})$-generic and

$$\mathbf{V}[G] = \mathbf{V}[(G \ \ \mathcal{B}) \times (G \ \ \mathcal{B}^{\mathrm{c}})] = \mathbf{V}[G \ \ \mathcal{B}][G \ \ \mathcal{B}^{\mathrm{c}}].$$

Two-step iteration of a forcing notion P and a P-name for a forcing notion $\dot{\mathsf{Q}}$ is written as $\mathsf{P} * \dot{\mathsf{Q}}$. An iteration of length γ is written as $\mathsf{P}_\gamma = \langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha \mid \alpha \in \gamma\rangle$, where P_0 is the trivial forcing notion and each $\dot{\mathsf{Q}}_\alpha$ is a P_α-name such that $\ \ _{\mathsf{P}_\alpha}$ "$\dot{\mathsf{Q}}_\alpha$ is a forcing notion". If $\alpha < \gamma$, then the initial part of a condition $p \in \mathsf{P}_\gamma$ is written as $p \ \ \alpha$, and (the P_α-name for) the $\dot{\mathsf{Q}}_\alpha$-condition at index α is written as $p(\alpha)$. Similarly we define $G \ \ \alpha$ and $G(\alpha)$ for any $G \subseteq \mathsf{P}_\gamma$.

Preservation Theorems

Preserving (strategic) κ-closure under iterations and products is fairly straightforward, and follows from considering the forcing notion elementwise.

Theorem 4.1.15 — *Folklore, see e.g. [Cum10, Proposition 7.9]*
If κ is regular, then iterations or products with $<\kappa$- or $\leq\kappa$-support of (strategically) $<\kappa$-closed forcing notions are (strategically) $<\kappa$-closed.

[1] We will not use products with other supports in this dissertation.

Let $\overline{\mathsf{P}} = \prod_{\xi\in\mathcal{A}}^{\leq\kappa} \mathsf{P}_\xi$ be a $\leq\kappa$-support product of forcing notions that are $<\kappa$-closed and let $\langle p_\alpha \mid \alpha \in \gamma\rangle$ be a descending sequence of conditions in $\overline{\mathsf{P}}$ with $\gamma < \kappa$. If we write $\bigwedge_{\alpha\in\gamma} p_\alpha(\xi)$ for a canonical condition $p' \in \mathsf{P}_\xi$ such that $p' \leq p_\alpha(\xi)$ for each $\alpha \in \gamma$ (for example p' could be the greatest lower bound of $\{p_\alpha(\xi) \mid \alpha \in \gamma\}$, assuming it exists), then we can define:

$$\bigwedge_{\alpha\in\gamma} p_\alpha : \xi \mapsto \bigwedge_{\alpha\in\gamma} p_\alpha(\xi) \quad \text{for } \xi \in \mathcal{A}.$$

By the above Theorem 4.1.15, it follows that $\bigwedge_{\alpha\in\gamma} p_\alpha$ is a condition in $\overline{\mathsf{P}}$.

Preservation of chain conditions by iterations or products does not happen as nicely with uncountable support and needs to be treated with some care.

For products in the classical case, the preservation of $<\omega_1$-c.c. (or c.c.c.) is shown using a Δ-system lemma, originally discovered by Shanin [Sha46] in the context of topology. A family $\mathcal{X}$ is called a *Δ-system* if there exists a set r such that $x \cap y = r$ for all distinct $x, y \in \mathcal{X}$.

Lemma 4.1.16 — *See e.g. [Kun11, Lemma III.6.15]*
Assume $2^{<\kappa} = \kappa < \lambda$ with λ regular. Let $\mathcal{X}$ be a family of sets with $|x| < \kappa$ for all $x \in \mathcal{X}$ and $|\mathcal{X}| \geq \lambda$, then there exists $\mathcal{Y} \in [\mathcal{X}]^\lambda$ such that $\mathcal{Y}$ is a Δ-system.

The Δ-system lemma can be used to prove the following theorem, which we will require later in Chapters 5 and 6:

Theorem 4.1.17 — *Folklore, see e.g. [Jec86, Part I Theorem 4.12]*
If $2^\kappa = \kappa^+$, $|\mathsf{P}_\xi| \leq \kappa^+$ for each $\xi \in \mathcal{A}$, then the $\leq\kappa$-support product $\prod_{\xi\in\mathcal{A}}^{\leq\kappa} \mathsf{P}_\xi$ is $<\kappa^{++}$-c.c.

For iteration, the preservation of $<\kappa^+$-c.c. is guaranteed by finite support iteration, but is quite complex for higher supports. On the other hand, stronger properties, such as the above-mentioned $(\kappa, {<}\kappa)$-centredness, can be preserved if we make some additional assumptions to deal with limit cases, which do not exist in the classical preservation of σ-centred forcing notions.[2]

Definition 4.1.18
Let P be a $<\kappa$-closed and $(\kappa, {<}\kappa)$-centred forcing notion and let $\mathsf{P} = \bigcup_{\gamma\in\kappa} P_\gamma$ be the decomposition into $<\kappa$-linked sets. We say that P is $(\kappa, {<}\kappa)$-centred *with canonical lower bounds* if there exists some $f : {}^{<\kappa}\kappa \to \kappa$ such that for any $\lambda < \kappa$ and decreasing sequence $\langle p_\alpha \mid \alpha \in \lambda\rangle$, with γ_α such that $p_\alpha \in P_{\gamma_\alpha}$, there exists $p \in P_{f(\langle\gamma_\alpha \mid \alpha\in\lambda\rangle)}$ with $p \leq p_\alpha$ for all $\alpha \in \lambda$. $\triangleleft$

Theorem 4.1.19 — *[BBTFM18, Lemma 55]*
Let κ be regular uncountable and $2^{<\kappa} = \kappa$ and let P be an iteration of length $<(2^\kappa)^+$ with $<\kappa$-support of $<\kappa$-closed, $(\kappa, {<}\kappa)$-centred forcing notions with canonical lower bounds, such that all canonical lower bounds lie in the ground model. Then P is $<\kappa$-closed and $(\kappa, {<}\kappa)$-centred.

In many occasions, we wish to iterate with $<\kappa^+$-c.c. forcing notions that are not $(\kappa, {<}\kappa)$-centred, but for which we still wish to show that $<\kappa^+$-c.c. is preserved by iteration. We will use the following application of the Δ-system lemma in these cases.

[2] In our notation σ-centred means $(\omega, {<}\omega)$-centred.

Lemma 4.1.20 — *Folklore, see e.g. [BBTFM18, Lemma 56] for a comparable result*
Let $\mathsf{P} = \langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha \mid \alpha \in \lambda \rangle$ be a $<\kappa$-support iteration such that each P_α forces that $\dot{\mathsf{Q}}_\alpha$ is a (strategically) $<\kappa$-closed $<\kappa^+$-c.c. forcing notion with conditions of the form (s, x) such that $s \in {}^{<\kappa}\mathbf{V}$, and such that $(s, x) \parallel (s, x')$ for every $(s, x), (s, x') \in \dot{\mathsf{Q}}_\alpha$. Then P is $<\kappa^+$-c.c. ◁

Proof. Note that each P_α is (strategically) $<\kappa$-closed, hence $<\kappa$-distributive by Theorem 4.1.15, so for each condition $(s, x) \in (\mathsf{Q}_\alpha)^{\mathbf{V}^{\mathsf{P}_\alpha}}$ we have $s \in \mathbf{V}$. For $p \in \mathsf{P}$, since $p(\alpha)$ is a P_α-name with $p \Vdash_\alpha$ "$p(\alpha) = (\dot{s}, \dot{x}) \in \dot{\mathsf{Q}}_\alpha$", there exists $p' \leq p$ such that $p' \Vdash_\alpha$ "$\dot{s} = \check{t}$" for some $t \in \mathbf{V}$. If we let $p_0 = p$ and $p_{n+1} \leq p_n$ be such that for every $\alpha \in \operatorname{supp}(p_n)$ we have $p_{n+1} \Vdash_\alpha$ "$p_n(\alpha) = (\check{s}_\alpha, \dot{x}_\alpha)$" for some $s_\alpha \in \mathbf{V}$, then we may find p' with $p' \leq p_n$ for each $n \in \omega$ and $\operatorname{supp}(p') = \bigcup_{n \in \omega} \operatorname{supp}(p_n)$ such that $p' \Vdash_\alpha$ "$p'(\alpha) = (\check{s}_\alpha, \dot{x}_\alpha)$" for each $\alpha \in \operatorname{supp}(p')$.[3] We will call such p' ***decisive***, and it is clear by the above that the set of decisive conditions is dense in P.

Let $\mathcal{B} \in [\mathsf{P}]^{\kappa^+}$ be a set of decisive conditions and let $\mathcal{A} = \{\{(\alpha, s_\alpha^p) \mid \alpha \in \operatorname{supp}(p)\} \mid p \in \mathcal{B}\}$, where each s_α^p is the element of ${}^{<\kappa}\mathbf{V}$ forming the first coordinate of $p(\alpha)$, as decided by p. Since P is a $<\kappa$-support iteration, $\mathcal{A}$ is a family of sets of size $<\kappa$, so we can apply the Δ-system lemma on $\mathcal{A}$. It follows that there exists $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $|\mathcal{B}_0| = \kappa^+$ and for any $p, p' \in \mathcal{B}_0$ and $\alpha \in \operatorname{supp}(p) \cap \operatorname{supp}(p')$ we have $s_\alpha^p = s_\alpha^{p'}$. But then $p \parallel p'$ follows from our assumptions on P, thus $\mathcal{B}$ is not an antichain. Therefore P has the $<\kappa^+$-c.c. □

We will conclude this subsection with a general theorem on adding κ-reals by iteration.

Theorem 4.1.21 — *Folklore, see e.g. [Gol92, Lemma 1.20] for ${}^\omega\omega$*
Let $\mathsf{P} = \left\langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha \mid \alpha \in \gamma \right\rangle$ be a $<\kappa$- or $\leq\kappa$-support iteration and $\Vdash_{\mathsf{P}}$ "$\operatorname{cf}(\gamma) > \kappa$", then $({}^\kappa\kappa)^{\mathbf{V}^{\mathsf{P}}} = \bigcup_{\alpha \in \gamma} ({}^\kappa\kappa)^{\mathbf{V}^{\mathsf{P}_\alpha}}$, that is, for each P-name $\dot{f}$ such that $\Vdash_{\mathsf{P}}$ "$\dot{f} \in {}^\kappa\kappa$" there exists $\alpha \in \gamma$ such that $\dot{f}$ is equivalent to a P_α-name.

Fusion

For some of our forcing notions, particularly those forcing notions of Section 4.4 and Chapters 5 and 6, we wish to use a construction method known as ***fusion*** to have better control over the forcing conditions. We will present the notation for fusion in this section, as well as a generalisation of fusion that works for products of forcing notions.

Let P be a forcing notion, then a ***fusion ordering*** is a sequence $\langle \leq_\alpha \mid \alpha \in \kappa \rangle$ of relations on P such that:

- $q \leq_0 p$ iff $q \leq p$, and
- $q \leq_\beta p$ implies $q \leq_\alpha p$ for all $\alpha < \beta$.

A ***fusion sequence*** is a sequence of conditions $\langle p_\alpha \mid \alpha \in \kappa \rangle$ such that $p_\beta \leq_\alpha p_\alpha$ for all $\alpha \leq \beta \in \kappa$. We say that P is ***closed under fusion*** if every fusion sequence $\langle p_\alpha \mid \alpha \in \kappa \rangle$ has some p with $p \leq_\alpha p_\alpha$ for all $\alpha \in \kappa$.

[3] If Q_α is only strategically $<\kappa$-closed, we may need to use White's winning strategy to obtain suitable p_n such that a lower bound exists.

If $\overline{\mathsf{P}} = \prod^{\leq\kappa}_{\xi\in\mathcal{A}} \mathsf{P}_\xi$ is such that each P_ξ has a fusion ordering $\leq^\xi_\alpha$, then we define for each $p, q \in \overline{\mathsf{P}}$, $\alpha \in \kappa$, and $Z \subseteq \mathcal{A}$ the ***generalised fusion relation*** $q \leq_{Z,\alpha} p$ iff $q \leq_{\overline{\mathsf{P}}} p$ and for each $\xi \in Z$ we have $q(\xi) \leq^\xi_\alpha p(\xi)$.

A ***generalised fusion sequence*** is a sequence $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa\rangle$ such that:

(i) $p_\alpha \in \overline{\mathsf{P}}$ and $Z_\alpha \in [\mathcal{A}]^{<\kappa}$ for each $\alpha \in \kappa$,
(ii) $p_\beta \leq_{Z_\alpha,\alpha} p_\alpha$ and $Z_\alpha \subseteq Z_\beta$ for all $\alpha \leq \beta \in \kappa$,
(iii) for limit δ we have $Z_\delta = \bigcup_{\alpha\in\delta} Z_\alpha$,
(iv) $\bigcup_{\alpha\in\kappa} Z_\alpha = \bigcup_{\alpha\in\kappa} \mathrm{supp}(p_\alpha)$.

We call $\overline{\mathsf{P}}$ ***closed under generalised fusion*** if every generalised fusion sequence $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa\rangle$ has some $p \in \overline{\mathsf{P}}$ with $\mathrm{supp}(p) = \bigcup_{\alpha\in\kappa} \mathrm{supp}(p_\alpha)$ and $p \leq_{\alpha,Z_\alpha} p_\alpha$ for all $\alpha \in \kappa$. Note that point (iv) implies that for each $\xi \in \mathcal{A}$ we have $p(\xi) \leq_\alpha p_\alpha(\xi)$ for almost all $\alpha \in \kappa$.

4.2. Generic κ-Reals

Certain κ-reals from the extension may interact with the κ-reals from the ground model in peculiar combinatorial ways. Such generic κ-reals are closely connected with the cardinal characteristics we presented in Sections 2.4 and 3.2, and showing that certain forcing notions either add or do not add such κ-reals frequently forms the key to proving that two cardinal characteristics are consistently different.

We will describe various κ-reals in the following definition. Some κ-reals are elements of the bounded higher Baire space $\prod b$ and some are defined using slaloms, and require a parameter $h \in {}^\kappa\kappa$ as well.

Definition 4.2.1
Let $\mathbf{V} \subseteq \mathbf{W}$ be models of ZFC, for instance $\mathbf{W}$ could be a forcing extension of $\mathbf{V}$. Then we call a κ-real $f \in ({}^\kappa\kappa)^\mathbf{W}$:

- a ***dominating*** ***κ-real*** over $\mathbf{V}$ if $g \leq^* f$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$,
- an ***unbounded*** ***κ-real*** over $\mathbf{V}$ if $g \leq^\infty f$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$,
- an ***eventually di erent*** ***κ-real*** over $\mathbf{V}$ if $g =^\infty f$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$,
- a ***cofinally equal*** ***κ-real*** over $\mathbf{V}$ if $g =^\infty f$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$,
- an ***h-avoiding*** ***κ-real*** over $\mathbf{V}$ if $f \in^* \psi$ for all $\psi \in (\mathrm{Loc}^h_\kappa)^\mathbf{V}$,
- an ***h-antiavoiding*** ***κ-real*** over $\mathbf{V}$ if $f \in^\infty \psi$ for all $\psi \in (\mathrm{Loc}^h_\kappa)^\mathbf{V}$.

If $\varphi \in (\mathrm{Loc}^h_\kappa)^\mathbf{W}$, then we call φ:

- an ***h-localising*** κ-real over $\mathbf{V}$ if $g \in^* \varphi$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$,
- an ***h-antilocalising*** κ-real over $\mathbf{V}$ if $g \in^\infty \varphi$ for all $g \in ({}^\kappa\kappa)^\mathbf{V}$.

Let $b \in ({}^\kappa\kappa)^\mathbf{V}$. If $f \in (\prod b)^\mathbf{W}$, then we call f:

- a ***b-dominating*** ***κ-real*** over $\mathbf{V}$ if $g \leq^* f$ for all $g \in (\prod b)^\mathbf{V}$,

- a b-*unbounded κ-real* over $\mathbf{V}$ if $g \leq^\infty f$ for all $g \in (\prod b)^{\mathbf{V}}$,
- a b-*eventually different κ-real* over $\mathbf{V}$ if $g =^\infty f$ for all $g \in (\prod b)^{\mathbf{V}}$,
- a b-*cofinally equal κ-real* over $\mathbf{V}$ if $g =^\infty f$ for all $g \in (\prod b)^{\mathbf{V}}$,
- a (b,h)-*avoiding κ-real* over $\mathbf{V}$ if $f \in^* \psi$ for all $\psi \in (\mathrm{Loc}_\kappa^{b,h})^{\mathbf{V}}$,
- a (b,h)-*antiavoiding κ-real* over $\mathbf{V}$ if $f \in^\infty \psi$ for all $\psi \in (\mathrm{Loc}_\kappa^{b,h})^{\mathbf{V}}$.

Let $b, h \in ({}^\kappa\kappa)^{\mathbf{V}}$. If $\varphi \in (\mathrm{Loc}_\kappa^{b,h})^{\mathbf{W}}$, then we call φ:

- a (b,h)-*localising* κ-real over $\mathbf{V}$ if $g \in^* \varphi$ for all $g \in (\prod b)^{\mathbf{V}}$,
- a (b,h)-*antilocalising* κ-real over $\mathbf{V}$ if $g \in^\infty \varphi$ for all $g \in (\prod b)^{\mathbf{V}}$. ◁

Note that not all of these κ-reals have classical analogues in ${}^\omega\omega$. For instance, if $b \in {}^\omega\omega$, then the ground model ω-real $b-1$ will dominate all functions in $\prod b$, making the notion of a b-dominating ω-real trivial.

As with cardinal characteristics, these generic κ-reals are related to each other. Figure 4.1 gives an overview of those cases where the existence of a κ-real with property P implies the existence of a (possibly different) κ-real with property Q, expressed by an arrow $P \to Q$. This diagram could be compared to the higher Cichoń diagram and the diagram in Figure 3.1. We believe most arrows are clear on inspection, but let us give a brief explanation.

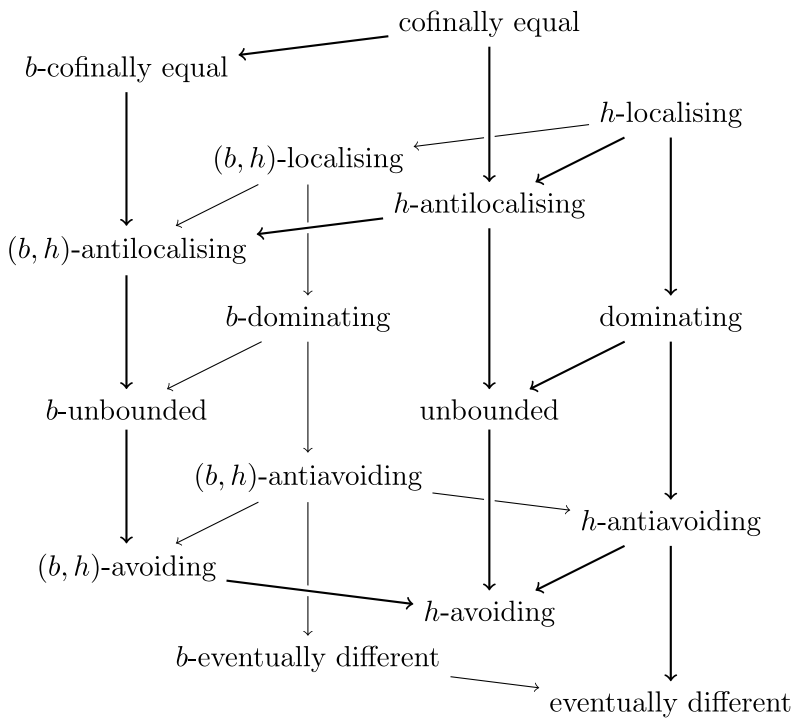


Figure 4.1: Implications between different types of κ-reals for a given fixed b and h with $h < \mathrm{cf}(b)$.

It is immediate that h-localising κ-reals are h-antilocalising, dominating κ-reals are unbounded, and h-antiavoiding κ-reals are h-avoiding.

If φ is h-localising, then $\sup(\varphi)$, that is, $\alpha \mapsto \sup(\varphi(\alpha))$ is dominating. If f is dominating, then $\sup(\psi) \leq^* f$ for all ground model h-slaloms ψ, hence f is h-antiavoiding, which is clearly eventually different by considering the singleton slalom $\varphi_g : \alpha \mapsto \{g(\alpha)\}$. The same reasoning applies for the bounded variants as long as we assume that $\mathrm{cf}(h) < b$, and for the dual implications, namely: “cofinally equal” implies “h-antilocalising” implies “unbounded” implies “h-avoiding”.

A cofinally equal f gives a b-cofinally equal f' by letting $f' : \alpha \mapsto 0$ when $\alpha \notin b(\alpha)$ and $f'(\alpha) = f(\alpha)$ otherwise. If φ is h-localising, then $\varphi \quad b : \alpha \mapsto \varphi(\alpha) \cap b(\alpha)$ is (b, h)-localising. Similarly $\varphi \quad b$ is (b, h)-antilocalising whenever φ is h-antilocalising.

Dually, if f is b-eventually different, it automatically is eventually different from any $g \in {}^\kappa\kappa$ since for $g(\alpha) \notin b(\alpha)$ we automatically have $g(\alpha) \neq f(\alpha)$ (since $f \in \prod b$). If f is (b, h)-antiavoiding, then $f \in^\infty (\psi \quad b)$ for every (unbounded) h-slalom ψ from the ground model, but since $f \in \prod b$ it is h-antiavoiding as well. Similarly a (b, h)-avoiding f is h-avoiding.

In the remainder of this chapter and in the open questions, we will try to answer the question whether Figure 4.1 is complete, that is, if there are other arrows that should be drawn, or if some of the arrows are reversible. Such observations largely coincide with proving independence results concerning our cardinal characteristics. We describe the general process with the following remark.

Remark 4.2.2

Let P be one of the properties of a κ-real, then P refers to a relation and we can consider $\mathsf{d}(P)$ and $\mathsf{b}(P)$ to be the norms of the corresponding relational system and its dual. Likewise, P has a dual property $P^\perp$. The following table gives an overview of the kinds of κ-reals and their corresponding cardinal characteristics.

P	$P^\perp$	relation	$\mathsf{d}(P)$	$\mathsf{b}(P)$
dominating	unbounded	$\leq^*$	$\mathsf{d}_\kappa(\leq^*)$	$\mathsf{b}_\kappa(\leq^*)$
eventually different	cofinally equal	$=^\infty$	$\mathsf{d}_\kappa(=^\infty)$	$\mathsf{b}_\kappa(=^\infty)$
h-localising	h-avoiding	$\in^*$	$\mathsf{d}^h_\kappa(\in^*)$	$\mathsf{b}^h_\kappa(\in^*)$
h-antiavoiding	h-antilocalising	$\ni^\infty$	$\mathsf{d}^h_\kappa(\ni^\infty)$	$\mathsf{b}^h_\kappa(\ni^\infty)$

Naturally, the same concepts apply to the bounded κ-reals, and their relations restricted to $\prod b$ and $\mathrm{Loc}^{b,h}_\kappa$. Let us consider two scenarios.

Scenario 1. Let P be a cardinal preserving $<\kappa$- or $\leq\kappa$-support iteration of length γ such that $_{\mathsf{P}}$ “$\mathrm{cf}(\gamma) > \kappa$” and a P κ-real is added cofinally often. Then $\mathbf{V}^{\mathsf{P}}$ “$\mathsf{d}(P) \leq \mathrm{cf}(\gamma) \leq \mathsf{b}(P)$”. Namely, if $D \subseteq \mathbf{V}^{\mathsf{P}}$ is a set of size $\mathrm{cf}(\gamma)$ consisting of P κ-reals that are added in cofinally many stages of the iteration, and $g \in \mathbf{V}^{\mathsf{P}}$, then g is added by an initial part of the iteration (Theorem 4.1.21), and thus one of the $f \in D$ is related to g by P. On the other hand, if $A \subseteq \mathbf{V}^{\mathsf{P}}$ is a set of κ-reals of size $< \mathrm{cf}(\gamma)$, then A is added by an initial part of the iteration, but then a later step adds a P κ-real that relates to all elements of A, so A is no witness for $\mathsf{b}(P)$.

Scenario 2. Let P be a cardinal preserving forcing notion that does not add P κ-reals, then $\mathbf{V}^{\mathsf{P}}$ "$\mathsf{b}(P) \leq (2^\kappa)^{\mathbf{V}}$". Namely, the set of all ground model κ-reals is a witness for $\mathsf{b}(P)$, since no new κ-real is related to all ground model κ-reals by P, or dually speaking, for every new κ-real there is a ground model real that is related to it by $P^\perp$. ◁

By answering the question whether Figure 4.1 is complete, we also show whether certain independence proofs are possible with the strategy described above. In the next subsection we will show how some of the properties from Section 4.1 may prevent certain kinds of κ-reals from being added. Some of the results we prove also extend Scenario 2, and give us a method to show that $\mathsf{d}(P)$ is not decreased by certain forcing notions that do not add P κ-reals.

Forcing Properties and κ-Reals

On ${}^\omega\omega$ one can show that σ-centred forcing do not add random reals, see e.g. [BJ95, Lemma 6.5.30]. Since there is no consensus over what a higher random forcing on ${}^\kappa\kappa$ would be, it is unclear what the general version of this lemma should be. The technique that is used is nevertheless quite versatile and can be used to prove several similar results. We will give two such results based on this method, linking $(\kappa, {<}\kappa)$-centred forcing notions to b-eventually different κ-reals, and (κ,κ)-calibre forcing notions to dominating κ-reals.

Lemma 4.2.3
If P is $(\kappa, {<}\kappa)$-centred and ${}_{\mathsf{P}}$ "$\dot f \in \prod b$", then there exists a family $\{f_\gamma \mid \gamma \in \kappa\} \subseteq \prod b$ in the ground model such that if $g =^\infty f_\gamma$ for all $\gamma \in \kappa$ then ${}_{\mathsf{P}}$ "$g =^\infty \dot f$". ◁

Proof. We let $\mathsf{P} = \bigcup_{\gamma\in\kappa} \mathsf{P}_\gamma$ such that each P_γ is ${<}\kappa$-linked. Given $\alpha, \gamma \in \kappa$, we define $f_\gamma(\alpha) = \min\{\beta \in b(\alpha) \mid \forall p \in \mathsf{P}_\gamma(p$ "$\dot f(\alpha) \neq \beta$")$\}$, then $f_\gamma \in \prod b$: if not, there exists $\alpha \in \kappa$ such that for each $\beta \in b(\alpha)$ there is some $p_\beta \in \mathsf{P}_\gamma$ such that p_β "$\dot f(\alpha) \neq \beta$". But then $\{p_\beta \mid \beta \in b(\alpha)\}$ has no common extension, contradicting that P_γ is ${<}\kappa$-linked.

Suppose that $g \in \prod b$ and $g =^\infty f_\gamma$ for all $\gamma \in \kappa$ and let $\alpha_0 \in \kappa$ and $p \in \mathsf{P}$ be arbitrary. There exists $\gamma \in \kappa$ such that $p \in \mathsf{P}_\gamma$, and since $g =^\infty f_\gamma$ we can find $\alpha \geq \alpha_0$ such that $g(\alpha) = f_\gamma(\alpha)$. But, by construction of f_γ we know that p "$\dot f(\alpha) \neq f_\gamma(\alpha)$", thus there exists $p' \leq p$ such that p' "$\dot f(\alpha) = g(\alpha)$". Since α_0 and p were arbitrary, we see that ${}_{\mathsf{P}}$ "$\dot f =^\infty g$". □

Corollary 4.2.4
If P is $(\kappa, {<}\kappa)$-centred then P does not add a b-eventually different κ-real.

Corollary 4.2.5
If P is $(\kappa, {<}\kappa)$-centred and preserves cardinals, then $\mathbf{V}^{\mathsf{P}}$ "$\mathsf{d}^b_\kappa(=^\infty) \geq (\mathsf{d}^b_\kappa(=^\infty))^{\mathbf{V}}$".

Lemma 4.2.6
If P is (κ,κ)-calibre and ${}_{\mathsf{P}}$ "$\dot f \in {}^\kappa\kappa$", then there exists a family $\{f_\gamma \mid \gamma \in \kappa\} \subseteq {}^\kappa\kappa$ in the ground model such that if $f_\gamma \leq^\infty g$ for all $\gamma \in \kappa$, then ${}_{\mathsf{P}}$ "$\dot f \leq^\infty g$". ◁

Proof. We let $\mathsf{P} = \bigcup_{\gamma\in\kappa} \mathsf{P}_\gamma$ such that each P_γ has calibre κ. For each $\gamma \in \kappa$ we define $f_\gamma(\alpha) = \min\{\beta \mid \forall p \in \mathsf{P}_\gamma(p$ "$\dot f(\alpha) \geq \beta$")$\}$, then $f_\gamma \in {}^\kappa\kappa$: if not, there exists $\alpha \in \kappa$ such that for each

$\beta \in \kappa$ there is some $p_\beta \in \mathsf{P}_\gamma$ with $p_\beta \quad$ “ $\dot{f}(\alpha) \geq \beta$ ”, where necessarily $|\{p_\beta \mid \beta \in \kappa\}| = \kappa$, since there exists no p such that $p \quad$ “ $\dot{f}(\alpha) \geq \beta$ ” for all $\beta \in \kappa$. Now P_γ has calibre κ, therefore there exists some $q \in \mathsf{P}$ with $q \leq p_\beta$ for all $\beta \in X \subseteq \kappa$ with $|X| = \kappa$, which means that $q \quad$ “ $\dot{f}(\alpha) \geq \beta$ ” for all $\beta \in \kappa$, a contradiction.

Suppose that $f_\gamma \leq^\infty g$ for all $\gamma \in \kappa$ and let $\alpha_0 \in \kappa$ and $p \in \mathsf{P}$ be arbitrary. There exists $\gamma \in \kappa$ such that $p \in \mathsf{P}_\gamma$ and since $f_\gamma \leq^\infty g$ there exists $\alpha \geq \alpha_0$ such that $g(\alpha) \geq f_\gamma(\alpha)$. Then $p \quad$ “ $\dot{f}(\alpha) \geq f_\gamma(\alpha)$ ”, thus there exists $p' \leq p$ such that $p' \quad$ “ $\dot{f}(\alpha) < f_\gamma(\alpha) \leq g(\alpha)$ ”. Since α_0 and p were arbitrary we see that $\quad_{\mathsf{P}}$ “ $\dot{f} \leq^\infty g$ ”. □

Corollary 4.2.7

If P is (κ, κ)-calibre, then P does not add a dominating κ-real.

Corollary 4.2.8

If P is (κ, κ)-calibre and preserves cardinals, then $\mathbf{V}^{\mathsf{P}} \quad$ “ $\mathsf{d}_\kappa(\leq^*) \geq (\mathsf{d}_\kappa(\leq^*))^{\mathbf{V}}$ ”.

There is a clear connection between ${}^\kappa\kappa$-bounding forcing notions and unbounded κ-reals.

Lemma 4.2.9

A forcing notion P is ${}^\kappa\kappa$-bounding if and only if P does not add an unbounded κ-real. ◁

Proof. If P is ${}^\kappa\kappa$-bounding, every name $\dot{f}$ such that $p \quad$ “ $\dot{f} \in {}^\kappa\kappa$ ” has some $q \leq p$ and $b \in {}^\kappa\kappa$ such that $q \quad$ “ $\dot{f} \in \prod b$ ”, hence $\dot{f}$ does not name an unbounded κ-real.

If P is not ${}^\kappa\kappa$-bounding, let $\dot{f}$ be a counterexample, that is, there exists p such that $p \quad$ “ $\dot{f} \in {}^\kappa\kappa$ ” and for all $q \leq p$ and $g \in {}^\kappa\kappa$, we have $q \quad$ “ $\dot{f} <^* g$ ”. Since this holds for all $q \leq p$ we have in fact $p \quad$ “ $\dot{f} <^* g$ ”, equivalently, $p \quad$ “ $g \leq^\infty \dot{f}$ ”. □

The h-Sacks and (b, h)-Laver properties provide a similar connection for h-avoiding and (b, h)-avoiding κ-reals respectively. These facts are proved exactly as in the above lemma.

Lemma 4.2.10

A forcing notion P has the h-Sacks property iff P does not add an h-avoiding κ-real.

Lemma 4.2.11

A forcing notion P has the (b, h)-Laver property iff P does not add a (b, h)-avoiding κ-real.

Finally, the h-Sacks property implies ${}^\kappa\kappa$-bounding by Lemma 4.1.14 and thus prevents unbounded κ-reals from being added. We can similarly show that the (b, h)-Laver property prevents b-unbounded κ-reals from being added.

Lemma 4.2.12

If a forcing notion P has the (b, h)-Laver property for some $h \in {}^\kappa\kappa$ with $\mathrm{cf}(h) < b$, then P does not add a b-unbounded κ-real. ◁

Proof. Suppose $\dot{f}$ and $p \in \mathsf{P}$ are such that $p \quad$ “ $\dot{f} \in \prod b$ ”, then let $\varphi \in \mathrm{Loc}_\kappa^{b,h}$ be such that $q \quad$ “ $\dot{f} \in^* \varphi$ ” for some $q \leq p$. Clearly $q \quad$ “ $\dot{f}(\alpha) \leq \sup(\varphi(\alpha))$ ” for almost all $\alpha \in \kappa$, and since $\mathrm{cf}(h) < b$ we see that $\sup(\varphi(\alpha)) \in b(\alpha)$ for all $\alpha \in \kappa$. Hence $\dot{f}$ is not b-unbounded. □

4.3. Forcing Notions with $<\kappa^+$-c.c.

In this section and the next we will define a range of forcing notions and their properties, and we investigate what kinds of κ-reals are added by each forcing notion. We also give some easy independence results concerning the cardinal characteristics of the previous chapters. The forcing notions that are treated in this section contain a part of the condition giving a κ-Cohen generic κ-real. This part also allows us to show that the condition is $<\kappa^+$-c.c.

In the next section we will consider forcing notions that are not $<\kappa^+$-c.c. Those forcing notions will be defined using perfect trees, some of which do not add a κ-Cohen generic.[4]

For many subsections we fix certain assumptions on κ and any parameters at the start of the subsection, to avoid having to mention them when stating results.

κ-Cohen Forcing

Assumptions. We assume that κ is regular uncountable.

Perhaps the most basic forcing notion that adds ω-reals is Cohen forcing, and with it, the most basic $<\kappa$-distributive forcing notion that adds κ-reals could be considered to be κ-Cohen forcing.

Definition 4.3.1

We define *κ-Cohen forcing* C_κ as the set of conditions ${}^{<\kappa}\kappa$, where $t \leq s$ if $s \subseteq t$. ◁

If $\mathbf{V} \subseteq \mathbf{W}$ are models of ZFC and there is a function $f \in ({}^{\kappa}\kappa)^{\mathbf{W}}$ such that $\{s \in (\mathsf{C}_\kappa)^{\mathbf{V}} \mid s \subseteq f\}$ is a generic filter, then we call f a *κ-Cohen generic (κ-real)*.[5] In other words, if $G \subseteq \mathsf{C}_\kappa$ is a generic filter over $\mathbf{V}$, then $\bigcup G$ is a κ-Cohen generic over $\mathbf{V}$.

Lemma 4.3.2 — *Folklore*

C_κ is $<\kappa$-closed. If $2^{<\kappa} = \kappa$, then C_κ is $(\kappa, <\kappa)$-centred. ◁

Proof. If κ is regular, and $\langle s_\alpha \mid \alpha \in \lambda\rangle$ is a descending sequence with $s_\alpha \in {}^{<\kappa}\kappa$ for each $\alpha \in \lambda$ and $\lambda < \kappa$, then $\bigcup_{\alpha\in\lambda} \mathrm{dom}(s_\alpha) \in \kappa$, and thus $\bigcup_{\alpha\in\lambda} s_\alpha \in {}^{<\kappa}\kappa$. Trivially, if $|\mathsf{C}_\kappa| = 2^{<\kappa} = \kappa$, then C_κ is $(\kappa, <\kappa)$-centred. □

Mirroring the situation on ${}^{\omega}\omega$, we see that κ-Cohen forcing has a special role in adding κ-reals for two reasons. If $\kappa = 2^{<\kappa}$, then it is the only nontrivial $<\kappa$-closed forcing notion of size κ that adds κ-reals, and due to this κ-Cohen generic reals are automatically added in limit stages of cofinality κ in $<\kappa$-support iterations of forcing notions. The latter means that if one wishes to avoid adding κ-Cohen generics, one has to resort to either products of forcing notions or to iterations of higher support.

[4] In fact, there is significant overlap between these two descriptions, namely, some of the perfect tree forcing notions do add a κ-Cohen generic, and all of the forcing notions in this section could be represented as forcing with perfect trees.

[5] One could call any set that codes the generic filter a κ-Cohen generic, but we reserve this terminology specifically for those κ-reals of the form $\bigcup G$ for some generic filter G $\quad \mathsf{C}_\kappa = {}^{<\kappa}\kappa$.

Definition 4.3.3

A forcing notion P is called ***separable*** if for each $p \in \mathsf{P}$ there are $q, r \in \mathsf{P}$ such that $q \perp r$ and $q \leq p$ and $r \leq p$. $\lhd$

Theorem 4.3.4 — *Folklore*

If P is a $<\kappa$-closed separative forcing notion and $|\mathsf{P}| = \kappa$, then P is forcing equivalent to C_κ. $\lhd$

Proof. We can construct for each $p_0 \in \mathsf{P}$ a strictly descending sequence $\langle p_\alpha \mid \alpha \in \kappa\rangle$ of conditions (using $<\kappa$-closure) and a sequence $\langle q_\alpha \mid \alpha \in \kappa\rangle$ such that $q_\alpha \perp p_{\alpha+1}$ for each $\alpha \in \kappa$ (using that P is separative). It then follows that $\{q_\alpha \mid \alpha \in \kappa\}$ is an antichain. Since $|\mathsf{P}| = \kappa$, we see that below any condition $p \in \mathsf{P}$ there exists a ***maximal*** antichain of size κ.

Let $\mathsf{C}^*_\kappa \subseteq \mathsf{C}_\kappa$ consist of all conditions $s \in \mathsf{C}_\kappa$ with $\mathrm{dom}(s)$ not a limit ordinal, then C^*_κ clearly densely embeds into C_κ. We now embed C^*_κ in P recursively.

We set $\pi(?) = \mathbb{1}_\mathsf{P}$. Given we have constructed $\pi(s)$ for some $s \in \mathsf{C}^*_\kappa$, we can enumerate a maximal antichain $\langle q^s_\alpha \mid \alpha \in \kappa\rangle$ below $\pi(s)$ in P. Then we define $\pi(s^\frown\langle\alpha\rangle) = q^s_\alpha$. Let $s \in \mathsf{C}_\kappa \setminus \mathsf{C}^*_\kappa$ be such that $\pi(s \quad \xi)$ has been defined for all $\xi \in \mathrm{dom}(s)$, then there exists a condition $p \in \mathsf{P}$ such that $p \leq \pi(s \quad \xi)$ for all $\xi \in \mathrm{dom}(s)$ by $<\kappa$-closure. Therefore, we can find a maximal antichain $\langle q^s_\alpha \mid \alpha\rangle$ of conditions that are below $\pi(s \quad \xi)$ for all $\xi \in \mathrm{dom}(s)$ and we define $\pi(s^\frown\langle\alpha\rangle) = q^s_\alpha$.

It is clear that π is an embedding, but to prove that π can be constructed to be dense, enumerate $\mathsf{P} = \{p_\beta \mid \beta \in \kappa\}$ and note that for each $\beta \in \kappa$ we may find $\xi \in \kappa$ and $s^\frown\langle\xi\rangle \in {}^{\beta+1}\kappa \subseteq \mathsf{C}^*_\kappa$ with $\pi(s^\frown\langle\xi\rangle) \leq p_\beta$: simply let the maximal antichain $\langle q^s_\alpha \mid \alpha \in \kappa\rangle$ be chosen such that $p_\beta \geq q^s_\alpha$ for some $\alpha \in \kappa$. $\square$

Corollary 4.3.5 — *Folklore*

The following forcing notions are forcing equivalent to C_κ:

(i) C^2_κ, which has the set of conditions ${}^{<\kappa}2$ ordered by $t \leq s$ iff $s \subseteq t$.

(ii) C^b_κ, which has the set of conditions $\prod_{<\kappa} b$ ordered by $t \leq s$ iff $s \subseteq t$.

(iii) Any product or iteration with $<\kappa$-support of C_κ of length $< \kappa^+$, assuming $2^{<\kappa} = \kappa$. $\lhd$

Proof. (i) and (ii) are clear.[6] For (iii), we argue firstly that iterations and products of κ-Cohen forcing with $<\kappa$-support are the same thing, as follows. The set of conditions ${}^{<\kappa}\kappa$ is absolute under κ-Cohen forcing extensions (which follows from $<\kappa$-distributivity), thus for any condition p_0 of the $<\kappa$-support iteration, we can construct a descending sequence $\langle p_n \mid n \in \omega\rangle$ such that for each $n \in \omega$ and $\alpha \in \mathrm{supp}(p_n)$ there is $s_\alpha \in {}^{<\kappa}\kappa$ for which $p_{n+1} \quad \alpha \quad$ "$p_n(\alpha) = \check{s}_\alpha$". Then let $p' \leq p_n$ for each $n \in \omega$ and $\mathrm{supp}(p') = \bigcup_{n\in\omega} \mathrm{supp}(p_n)$ (which is still of cardinality $< \kappa$), then $p' \quad \alpha$ decides the value of $p'(\alpha)$ for each $\alpha \in \mathrm{supp}(p')$. Hence p' is equivalent to a condition q in the product forcing, to be precise, $q : \alpha \mapsto s_\alpha$ if $\alpha \in \mathrm{supp}(p')$ and $q : \alpha \mapsto ?$ otherwise.

Secondly, if $\alpha < \kappa^+$, then the $<\kappa$-support product of α copies of $\mathsf{C}_\kappa = {}^{<\kappa}\kappa$ is isomorphic to a subset of the set of partial functions from $\alpha \times \kappa$ to κ with a domain of size $<\kappa$. There only exist $2^{<|\alpha\cdot\kappa|} = 2^{<\kappa} = \kappa$ many such partial functions. $\square$

[6] This could be compared to Lemma 3.1.1.

Theorem 4.3.6 — *Folklore, see e.g. [Gol92, Example 0.2] for* ${}^{\omega}\omega$

If $\mathsf{P} = \langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha \mid \alpha < \kappa \rangle$ is a $<\kappa$-support iteration such that $\mathsf{P}_\alpha \quad$ "$\dot{\mathsf{Q}}_\alpha \ni \dot{q}_\alpha, \dot{q}'_\alpha$ and $\dot{q}_\alpha \perp \dot{q}'_\alpha$" for each $\alpha \in \kappa$, then P adds a κ-Cohen generic. ◁

Proof. We consider the dense subset $\mathsf{P}^* \subseteq \mathsf{P}$ consisting of conditions p with $\operatorname{supp}(p) = \xi$ for some $\xi \in \kappa$. Let $\pi : p \mapsto s \in {}^{\xi}2$, where for each $\alpha \in \xi$ we let $s(\alpha) = 0$ if $p \quad \alpha \quad$ "$p(\alpha) \leq \dot{q}_\alpha$" and $s(\alpha) = 1$ otherwise. Then for any P-generic G it is easy to see that $\pi[G]$ is generic for κ-Cohen forcing with the set of conditions ${}^{<\kappa}2$. □

For κ-Cohen forcing, we know exactly which of the previously mentioned κ-reals are added, determined by the following two lemmas.

Lemma 4.3.7

C_κ adds a cofinally equal κ-real. ◁

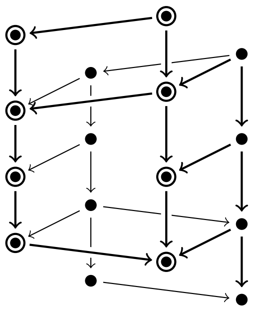
Proof. Fix some $g \in {}^{\kappa}\kappa$ and $s \in \mathsf{C}_\kappa$ and let $\dot{f}$ name the κ-Cohen generic κ-real. For any $\alpha_0 \in \kappa$ there exists $\alpha \geq \alpha_0$ and $t \in \mathsf{C}_\kappa$ such that $\alpha \in \operatorname{dom}(t)$ and $t(\alpha) = g(\alpha)$, or in other words, $t \quad$ "$\dot{f}(\alpha) = g(\alpha)$". It follows from genericity that $\quad_{\mathsf{C}_\kappa}$ "$\dot{f} =^\infty g$". Since g was arbitrary, $\dot{f}$ names a cofinally equal κ-real. □

Lemma 4.3.8

If $2^{<\kappa} = \kappa$, then C_κ does not add an eventually different κ-real. ◁

Proof. The proof is a simple version of Lemma 4.2.3. Since $2^{<\kappa} = \kappa$, we can enumerate C_κ as $\{p_\alpha \mid \alpha \in \kappa\}$. Let $\dot{f}$ name a κ-real and define $g(\alpha) = \min\{\xi \in \kappa \mid p_\alpha \quad$ "$\dot{f}(\alpha) \neq \xi$"$\}$ for each $\alpha \in \kappa$. Since $p_\alpha \quad$ "$\dot{f}(\alpha) \in \kappa$", we see that $g \in {}^{\kappa}\kappa$.

Let $s \in \mathsf{C}_\kappa$ and $\alpha_0 \in \kappa$. Then there exists $\alpha \geq \alpha_0$ such that $p_\alpha \leq s$. By definition of g we have $p_\alpha \quad$ "$\dot{f}(\alpha) \neq g(\alpha)$", hence there exists some $t \leq p_\alpha$ such that $t \quad$ "$\dot{f}(\alpha) = g(\alpha)$". It follows from genericity that $\quad_{\mathsf{C}_\kappa}$ "$\dot{f} =^\infty g$", and thus $\dot{f}$ is not an eventually different κ-real. □



A schematic representation of Figure 4.1, showing the effect of κ-Cohen forcing.

Legend
◉ added by the forcing
• not added by the forcing

We define the κ-*Cohen model* as the result of forcing with a $<\kappa$-support iteration of κ^{++} copies of C_κ over a ground model $\mathbf{V} \quad$ "$2^\kappa = \kappa^+$". We can show that there exist no eventually different κ-reals over $\mathbf{V}$ in the κ-Cohen model, the proof of which comes down to the well-known property that any new κ-real added by the entire iteration is actually in the generic extension of a single κ-Cohen forcing over the ground model.

Theorem 4.3.9
Let $2^{<\kappa} = \kappa$ and let $\mathbf{V}$ "$2^\kappa = \kappa^+$". If $\mathbf{W}$ is the κ-Cohen model over $\mathbf{V}$, then $\mathbf{W}$ does not contain eventually different κ-reals over $\mathbf{V}$. $\triangleleft$

Proof. Let $\mathsf{P} = \langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha \mid \alpha \in \kappa^{++} \rangle$ with $_{\mathsf{P}_\alpha}$ "$\dot{\mathsf{Q}}_\alpha = \mathsf{C}_\kappa$" be the $<\kappa$-support iteration of the κ-Cohen model, and let $\dot{f}$ be a P-name such that $_{\mathsf{P}}$ "$\dot{f} \in {}^\kappa\kappa$". Remember by Theorem 4.1.21 $<\kappa$-support iterations do not add new κ-reals at limit steps of cofinality $>\kappa$, thus $\dot{f}$ is added by an initial segment of the iteration, say by P_α.

By Lemma 4.1.20, we see that P is $<\kappa^+$-c.c.

Remember from the proof of Corollary 4.3.5 that we may consider P_α as a product forcing with $<\kappa$-support instead. Consider a ***nice*** P_α-name for $\dot{f}$, that is, a name containing only elements of the form $(\langle \check{\gamma}, \check{\beta} \rangle, p)$ where $\langle \check{\gamma}, \check{\beta} \rangle$ is the canonical name for the pair of ordinals $(\gamma, \beta) \in \kappa \times \kappa$ and $p \in \mathsf{P}_\alpha$, such that for any two distinct elements of the name $(\langle \check{\gamma}, \check{\beta} \rangle, p)$ and $(\langle \check{\gamma}', \check{\beta}' \rangle, p')$ with $\gamma = \gamma'$ we have $p \perp p'$.

For each $\gamma \in \kappa$ the set $A_\gamma = \{p \in \mathsf{P}_\alpha \mid \exists \beta((\langle \check{\gamma}, \check{\beta} \rangle, p) \in \dot{f})\}$ is an antichain and hence has cardinality at most κ. Let $A = \bigcup_{\gamma \in \kappa} \bigcup_{p \in A_\gamma} \operatorname{supp}(p)$, then $|A| \leq \kappa$ since P_α is a $<\kappa$-support product. Therefore the nice name $\dot{f}$ contains only conditions whose support are contained in A, and thus $\dot{f}$ is a $\prod_{\xi \in A}^{<\kappa} \mathsf{C}_\kappa$-name. But by Corollary 4.3.5 it follows that $\dot{f}$ is added by a single κ-Cohen forcing, and thus cannot be eventually different by Lemma 4.3.8. $\square$

Corollary 4.3.10 — *Folklore*
If κ is inaccessible, then $\operatorname{non}(\mathcal{M}_\kappa) = \kappa^+ < \kappa^{++} = \operatorname{cov}(\mathcal{M}_\kappa)$ holds in the κ-Cohen model. $\triangleleft$

Proof. Since the κ-Cohen model contains κ^{++} many cofinally equal κ-reals over $\mathbf{V}$, it follows that $\mathfrak{b}_\kappa(=^\infty) = \kappa^{++}$ in the κ-Cohen model. On the other hand, the κ-Cohen model contains no new eventually different κ-reals, and thus $\mathfrak{d}_\kappa(=^\infty) = (2^\kappa)^{\mathbf{V}} = \kappa^+$. The result then follows from Fact 2.5.4. $\square$

Note that the value of all other cardinal characteristics we have discussed so far are also decided in the κ-Cohen model by the above. We mention that an iteration of arbitrary length $\lambda > \kappa^{++}$ proves $\operatorname{non}(\mathcal{M}_\kappa) < \operatorname{cov}(\mathcal{M}_\kappa) = \lambda$ and that the above corollary also holds for the weaker assumption that $2^{<\kappa} = \kappa$, where we need to argue with the meagre ideal directly, see for example [Bre22, Section 3].

κ-Hechler Forcing

Assumptions. We assume that κ is regular uncountable.

Classically, Hechler forcing, also known as dominating real forcing, is a forcing notion that adds both dominating reals and Cohen reals. The same happens in κ, where we can generalise Hechler forcing to κ-Hechler forcing, first studied by Cummings & Shelah [CS95]. We describe the κ-Hechler model and the dual κ-Hechler model, which have both been previously discussed in [BBTFM18, Section 4.2] to investigate the higher Cichoń diagram.

Definition 4.3.11

We define κ-*Hechler forcing* D_κ to have conditions $p = (s, f)$ where $s \in {}^{<\kappa}\kappa$ and $f \in {}^{\kappa}\kappa$, where the ordering is defined as $(t, g) \leq (s, f)$ iff $s \subseteq t$ and $f(\alpha) \leq g(\alpha)$ for all $\alpha \in \kappa \setminus \mathrm{dom}(s)$ and $f(\alpha) \leq t(\alpha)$ for all $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$. ◁

If $\mathbf{V} \subseteq \mathbf{W}$ are models of ZFC and there exists $f \in ({}^{\kappa}\kappa)^{\mathbf{W}}$ such that $\{(s, g) \in (\mathsf{D}_\kappa)^{\mathbf{V}} \mid s \subseteq f\}$ is a $(\mathsf{D}_\kappa)^{\mathbf{V}}$-generic filter over $\mathbf{V}$, then we call f a κ-*Hechler generic (κ-real)*. Of the κ-Hechler condition (s, g), we can consider s to be an approximation of the κ-Hechler generic f, and g as a promise that $g \leq^* f$.

Lemma 4.3.12 — *See e.g. [BBTFM18, Section 4.2]*

D_κ adds a dominating κ-real and a κ-Cohen generic. ◁

Proof. For any condition (s, g) and $h \in {}^{\kappa}\kappa$ in the ground model there exists some condition $(s, h') \leq (s, g)$ with $h' \geq h$. If $\dot{f}$ names the κ-Hechler generic κ-real, then (s, h') “ $h \leq h' \leq^* \dot{f}$ ”, hence $\dot{f}$ names a dominating κ-real over the ground model.

Given the κ-Hechler generic f, let $f'(\alpha) = 0$ if $f(\alpha)$ is even[7] and $f'(\alpha) = 1$ otherwise. One can easily see that f' is generic for κ-Cohen forcing C^2_κ from Corollary 4.3.5. □

Lemma 4.3.13 — *[CS95, Lemma 7]*

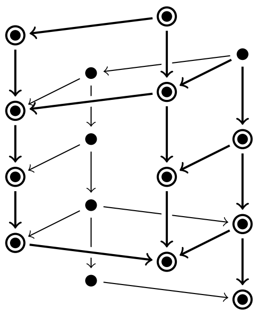
D_κ is $<\kappa$-closed and if $2^{<\kappa} = \kappa$, then D_κ is $(\kappa, <\kappa)$-centred with canonical bounds. ◁

Proof. Let $\lambda < \kappa$ and $\langle (s_\alpha, f_\alpha) \mid \alpha \in \lambda \rangle$ be a descending sequence of conditions, and define $s = \bigcup_{\alpha \in \lambda} s_\alpha$ and $f : \xi \mapsto \sup\{f_\alpha(\xi) \mid \alpha \in \lambda\}$, then $(s, f) \in \mathsf{D}_\kappa$ by regularity of κ. By the same argument $D_s = \{s\} \times {}^{\kappa}\kappa$ is a $<\kappa$-linked subset of D_κ, so with $2^{<\kappa} = \kappa$ it follows that $\mathsf{D}_\kappa = \bigcup_{s \in {}^{<\kappa}\kappa} D_s$ is $(\kappa, <\kappa)$-centred and that the ground model function witnessing the canonical bound is simply the function $\langle s_\alpha \mid \alpha \in \lambda \rangle \mapsto \bigcup_{\alpha \in \lambda} s_\alpha$. □

Corollary 4.3.14

If $2^{<\kappa} = \kappa$, then D_κ does not add a b-eventually different κ-reals for any $b \in {}^{\kappa}\kappa$. ◁

Proof. By Lemma 4.2.3. □



A schematic representation of Figure 4.1, showing the effect of κ-Hechler forcing, with $2^{<\kappa} = \kappa$.

Legend
- ◉ added by the forcing
- • not added by the forcing

We define the κ-*Hechler model* as the result of forcing with a $<\kappa$-support iteration of κ^{++} copies of D_κ over a ground model $\mathbf{V}$ “ $2^\kappa = \kappa^+$ ”. We define the ***dual*** κ-***Hechler model*** as the result

[7] We call an ordinal α *even* if $\alpha = \gamma + n$ for γ a limit ordinal and n 2 ω even

of forcing with a $<\kappa$-support iteration of $\kappa^{++}\cdot\kappa^+$ copies of D_κ (over $\mathbf{V}$ “$2^\kappa=\kappa^+$”). This is not the standard definition of the dual κ-Hechler model, but will yield a stronger result where we can prove values for $\mathfrak{d}^b_\kappa(=^\infty)$ and $\mathfrak{d}^h_\kappa(\in^*)$ not just for parameters b,h in the ground model, but also for those added by the forcing.

Theorem 4.3.15

If κ is inaccessible, then $\mathfrak{b}^b_\kappa(=^\infty)=\mathfrak{b}^h_\kappa(\in^*)=\kappa^+<\kappa^{++}=\mathrm{add}(\mathcal{M}_\kappa)$ holds in the κ-Hechler model for any $b,h\in{}^\kappa\kappa$ (including new κ-reals that are added by the forcing) for which the respective cardinal characteristics are nontrivial[8]. ◁

Proof. Let $\mathsf{P}=\langle\mathsf{P}_\alpha,\dot{\mathsf{Q}}_\alpha\mid\alpha\in\kappa^{++}\rangle$ be a $<\kappa$-support iteration, where $\dot{\mathsf{Q}}_\alpha$ is a P_α-name for κ-Hechler forcing D_κ and let $\mathbf{V}$ “$2^\kappa=\kappa^+$”, then $\mathbf{V}^{\mathsf{P}}$ is the κ-Hechler model.

Since every new κ-real is added by an initial stage of the iteration, for any $b\in({}^\kappa\kappa)^{\mathbf{V}^{\mathsf{P}}}$ there exists some $\alpha\in\kappa^{++}$ such that $b\in({}^\kappa\kappa)^{\mathbf{V}^{\mathsf{P}_\alpha}}$. Note that $2^\kappa=\kappa^+$ holds in $\mathbf{V}^{\mathsf{P}_\alpha}$ for any $\alpha<\kappa^{++}$, and thus $|(\prod b)^{\mathbf{V}^{\mathsf{P}_\alpha}}|=\kappa^+$. We claim that $(\prod b)^{\mathbf{V}^{\mathsf{P}_\alpha}}$ is a witness for $\mathfrak{b}^b_\kappa(=^\infty)=\kappa^+$. Let us assume[9] for notational convenience that in fact $b\in\mathbf{V}$.

Let $\dot f$ be a P-name such that $_{\mathsf{P}}$ “$\dot f\in\prod b$”, then there is some $\alpha<\kappa^{++}$ such that $f\in\mathbf{V}^{\mathsf{P}_\alpha}$, and thus we can assume that $\dot f$ is a P_α-name. Note that $\mathbf{V}$ “$\alpha<(2^\kappa)^+=\kappa^{++}$”, therefore P_α is $(\kappa,<\kappa)$-centred by Theorem 4.1.19, and thus $\dot f$ does not name a b-eventually different κ-real by Lemma 4.2.3. Since h-localising κ-reals imply the existence of b-eventually different κ-reals, no h-localising κ-reals are added either for any $h\in({}^\kappa\kappa)^{\mathbf{V}^{\mathsf{P}}}$.

In the extension, it follows that $(\prod b)^{\mathbf{V}}$ and $({}^\kappa\kappa)^{\mathbf{V}}$ witness that $\mathfrak{b}^b_\kappa(=^\infty)=\kappa^+$ and $\mathfrak{b}^h_\kappa(\in^*)=\kappa^+$ respectively. Note that each stage of the iteration adds a dominating κ-real, which implies $\mathfrak{b}_\kappa(\leq^*)=\kappa^{++}$, and a cofinally equal κ-real (a κ-Cohen generic), which implies $\mathfrak{d}_\kappa(=^\infty)=\kappa^{++}$. That $\mathrm{add}(\mathcal{M}_\kappa)=\kappa^{++}$ then follows from Facts 2.5.4 and 2.5.9. □

Theorem 4.3.16

If κ is inaccessible, then $\mathrm{cof}(\mathcal{M}_\kappa)=\kappa^+<\kappa^{++}=\mathfrak{d}^b_\kappa(=^\infty)=\mathfrak{d}^h_\kappa(\in^*)$ holds in the dual κ-Hechler model for any $b,h\in{}^\kappa\kappa$ (including new κ-reals that are added by the forcing) for which the respective cardinal characteristics are nontrivial. ◁

Proof. Let $\mathsf{P}=\langle\mathsf{P}_\alpha,\dot{\mathsf{Q}}_\alpha\mid\alpha\in\kappa^{++}\cdot\kappa^+\rangle$ be a $<\kappa$-support iteration, where $\dot{\mathsf{Q}}_\alpha$ is a P_α-name for κ-Hechler forcing D_κ and let $\mathbf{V}$ “$2^\kappa=\kappa^+$” and let G be a P-generic filter over $\mathbf{V}$, then $\mathbf{V}[G]$ is the dual κ-Hechler model.

Let $b,h\in({}^\kappa\kappa)^{\mathbf{V}[G]}$, then there exists some $\alpha\in\kappa^{++}\cdot\kappa^+$ such that $b,h\in({}^\kappa\kappa)^{\mathbf{V}[G_\alpha]}$. Let $\beta=\alpha+\kappa^{++}$, then $\beta<\kappa^{++}\cdot\kappa^+$ and $\mathbf{V}[G_\beta]$ “$\mathrm{add}(\mathcal{M}_\kappa)=\kappa^{++}=2^\kappa$” by similar arguments as in Theorem 4.3.15. This implies particularly that $\mathbf{V}[G_\beta]$ “$\mathfrak{d}^b_\kappa(=^\infty)=\mathfrak{d}^h_\kappa(\in^*)=\kappa^{++}=2^\kappa$” as well. We will work in $\mathbf{V}[G_\beta]$ as our new ground model. Let $\mathsf{P}'=\mathsf{P}/G_\beta\in\mathbf{V}[G_\beta]$ be the quotient

[8] In the sense of Section 3.4.

[9] We can do this without loss of generality, since the remainder of the forcing $\mathsf{P}/\mathsf{P}_\alpha$ is equivalent to the entire forcing interpreted in $\mathbf{V}^{\mathsf{P}_\alpha}$.

forcing and H be P'-generic such that $\mathbf{V}[G] = \mathbf{V}[G_\beta][H]$. Note that P' is itself (equivalent to) a $<\kappa$-support iteration of D_κ of length $\kappa^{++} \cdot \kappa^+$.

Since $\mathbf{V}[G_\beta]$ "$2^\kappa = \kappa^{++}$", we see that P' is $(\kappa, <\kappa)$-centred by Theorem 4.1.19. Therefore P' does not add b-eventually different (or h-localising) κ-reals, and consequently $\mathsf{d}^b_\kappa(=^\infty) = \mathsf{d}^h_\kappa(\in^*) = \kappa^{++}$ holds in the extension, by Corollary 4.2.5. On the other hand, each step of the iteration adds a dominating and cofinally equal κ-real, and the iteration has cofinality κ^+. Taking a cofinal sequence of length κ^+ of such dominating and cofinally equal κ-reals will form witnesses for $\mathsf{d}_\kappa(\leq^*) = \mathsf{b}_\kappa(=^\infty) = \kappa^+$. That $\mathrm{cof}(\mathcal{M}_\kappa) = \kappa^+$ then follows from Facts 2.5.4 and 2.5.9. □

Bounded κ-Hechler Forcing

Assumptions. We assume that κ is inaccessible. We will also assume that $b \in {}^\kappa\kappa$ is such that case (iii) of Theorem 3.4.2 is satisfied, that is, there exists a club set C such that for each $\xi \in C$ we have $\mathrm{cf}(b(\alpha)) > \xi$ for all $\alpha \geq \xi$, or equivalently, $\mathrm{cf}(b)$ is increasing and discontinuous on C.

In order to influence the bounded cardinal characteristics and κ-reals, we will define a bounded version of κ-Hechler forcing. Such bounded κ-Hechler forcing notions have been considered by others before as well, and form a key part in Shelah's [She20] proof of the consistency of $\mathrm{cov}(\mathcal{M}_\kappa) < \mathsf{d}_\kappa(\leq^*)$. We will use bounded κ-Hechler forcing to prove that $\kappa^+ < \mathsf{b}^b_\kappa(\leq^*)$ and $\mathsf{d}^b_\kappa(\leq^*) < 2^\kappa$ are consistent under the assumptions of case (iii) of Theorem 3.4.2. Finally we will show that bounded κ-Hechler forcing does not add dominating κ-reals if we assume κ is weakly compact.

Definition 4.3.17

We define b-κ-*Hechler forcing* D^b_κ to have conditions (s, f) with $s \in \prod_{<\kappa} b$ and $f : \kappa \setminus \mathrm{dom}(s) \to \kappa$ such that $f(\alpha) \in b(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$, where the ordering is defined as $(t, g) \leq (s, f)$ iff $s \subseteq t$ and $f(\alpha) \leq g(\alpha)$ for all $\alpha \in \mathrm{dom}(g)$ and $f(\alpha) \leq t(\alpha)$ for all $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$. ◁

It is clear from the definition, that D^b_κ adds a b-dominating κ-real, and for similar reasons as with κ-Hechler forcing D_κ, bounded κ-Hechler forcing D^b_κ also adds a κ-Cohen generic.

Note that D^b_κ is not $<\kappa$-closed in general. For instance, if we let $\overline{\alpha}$ be the constant function with value α, then $\langle (\mathbf{?}, \overline{\alpha}) \mid \alpha \in b(0) \rangle$ does not have a lower bound.

Lemma 4.3.18

D^b_κ is strategically $<\kappa$-closed and $<\kappa^+$-c.c. ◁

Proof. For strategic $<\kappa$-closure, at stage α of the game $\mathcal{G}(\mathsf{D}^b_\kappa, p)$, let $\langle (s_\xi, f_\xi) \mid \xi < \alpha \rangle$ and $\langle (s'_\xi, f'_\xi) \mid \xi < \alpha \rangle$ be the sequences of previous moves by White and Black respectively. The winning strategy for White will be to choose (s_α, f_α) such that $\alpha \leq \mathrm{dom}(s_\alpha) \in C$ in successor stages. Under this strategy, if α is a limit ordinal, then we have $\xi \leq \mathrm{dom}(s_\xi) \in C$ for each $\xi \in \alpha$. It follows that $\alpha \leq \mathrm{dom}(s_\alpha)$ and hence $\mathrm{cf}(b(\beta)) > \alpha$ for all $\beta \geq \mathrm{dom}(s_\alpha)$ by the assumptions on b and C. Therefore $\left\langle f'_\xi(\beta) \mid \xi \in \alpha \right\rangle$ is not cofinal in $b(\beta)$, thus we can define $f_\alpha : \beta \mapsto \bigcup_{\xi \in \alpha} f'_\xi(\beta)$ for each $\beta \in \kappa \setminus \mathrm{dom}(s_\alpha)$.

For $<\kappa^+$-c.c., note that for any $(s,f),(s,g) \in \mathsf{D}^b_\kappa$ we can choose $h(\alpha) = \max\{f(\alpha), g(\alpha)\}$ for all $\alpha \in \kappa \setminus \mathrm{dom}(s)$ to see that have $(s,h) \leq (s,f)$ and $(s,h) \leq (s,g)$. Thus, if $\mathcal{A} \subseteq \mathsf{D}^b_\kappa$ is an antichain and $(s,f),(t,g) \in \mathcal{A}$ are distinct, then we must have $s \neq t$, hence $|\mathcal{A}| \leq \kappa$. □

The above lemma does not only show that D^b_κ is $<\kappa^+$-c.c., but also that Lemma 4.1.20 applies, and thus that $<\kappa$-support iteration of D^b_κ is also $<\kappa^+$-c.c. This implies especially that iteration will not collapse cardinals or destroy the inaccessibility of κ.

Theorem 4.3.19

It is consistent that $\mathfrak{b}^b_\kappa(\leq^*) > \kappa^+$ and that $\mathfrak{d}^b_\kappa(\leq^*) < 2^\kappa$. ◁

Proof. Let $\mathsf{P} = \langle \mathsf{P}_\alpha, \dot{\mathsf{Q}}_\alpha \mid \alpha \in \kappa^{++} + \kappa^+ \rangle$ be a $<\kappa$-support iteration with $\mathsf{P}_\alpha \quad$ "$\dot{\mathsf{Q}}_\alpha = \dot{\mathsf{D}}^b_\kappa$" for each α and let $\mathbf{V} \quad$ "$2^\kappa = \kappa^+$". If $B \subseteq (\prod b)^{\mathbf{V}^{\mathsf{P}_{\kappa^{++}}}}$ is such that $|B| \leq \kappa^+$, then $B \in \mathbf{V}^{\mathsf{P}_\alpha}$ for some $\alpha < \kappa^{++}$, since the κ^{++}-th stage of the iteration does not add any κ-reals. The generic b-dominating κ-real that is added in the $\alpha+1$-th stage then dominates all elements of B, hence B is not unbounded in $\prod b$. This shows that $\mathbf{V}^{\mathsf{P}_{\kappa^{++}}} \quad$ "$\mathfrak{b}^b_\kappa(\leq^*) > \kappa^+$".

Next, let D consist of the b-dominating κ-reals that are added in the final κ^+ stages of the iteration, then D clearly forms a dominating family of size κ^+ in $\mathbf{V}^{\mathsf{P}}$, and it is easy to see that $\mathbf{V}^{\mathsf{P}} \quad$ "$2^\kappa = \kappa^{++}$", showing that $\mathbf{V}^{\mathsf{P}} \quad$ "$\mathfrak{d}^b_\kappa(\leq^*) < 2^\kappa$". □

Lemma 4.3.20 — *This proof is due to Jörg Brendle, private communication*

If κ is weakly compact, then D^b_κ has (κ,κ)-calibre. ◁

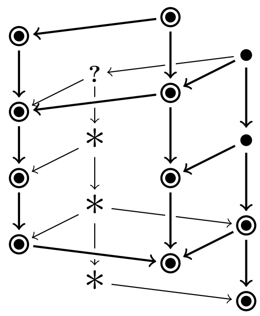
Proof. For any $s \in \prod_{<\kappa} b$ and $\{f_\alpha \mid \alpha \in \kappa\} \subseteq \prod b$, we will describe some $f \in \prod b$ and $A \in [\kappa]^\kappa$ such that $f(\xi) \geq f_\alpha(\xi)$ for all $\xi \in \kappa \setminus \mathrm{dom}(s)$ and $\alpha \in A$. It is then clear that $(s,f) \leq (s,f_\alpha)$, and thus $D_s = \{s\} \times \prod b \subseteq \mathsf{D}^b_\kappa$ has κ-calibre and $\mathsf{D}^b_\kappa = \bigcup_{s \in \prod_{<\kappa} b} D_s$ has (κ,κ)-calibre.

We will assume without loss of generality that $f_\alpha \neq f_\beta$ and $s \subseteq f_\alpha$ for all distinct $\alpha, \beta \in \kappa$. We will define $T = \left\{t \in \prod_{<\kappa} b \mid \exists\alpha\exists\beta(\alpha \neq \beta \wedge t \subseteq f_\alpha \cap f_\beta)\right\}$. Note that T is a κ-tree, because $\left|\prod_{\xi<\alpha} b(\xi)\right| < \kappa$ by inaccessibility of κ. Since κ is weakly compact, there exists a branch $g \in [T]$. We will need the following property of g: for any $\alpha_0, \gamma \in \kappa$ there exists some $\alpha \geq \alpha_0$ such that $\gamma \subseteq \mathrm{dom}(f_\alpha \cap g)$.

Now let $\langle \gamma_\xi \mid \xi \in \kappa \rangle$ enumerate the club C and for each ξ find some α_ξ with $\gamma_\xi \subseteq \mathrm{dom}(f_{\alpha_\xi} \cap g)$ such that $\langle \alpha_\xi \mid \xi \in \kappa \rangle$ is strictly increasing. Let $A = \{\alpha_\xi \mid \xi \in \kappa\}$ and $f : \beta \mapsto \bigcup_{\xi \in \kappa} f_{\alpha_\xi}(\beta)$, then these are as needed. Note that $f(\beta) \in b(\beta)$, because if ξ is such that $\beta \in [\gamma_\xi, \gamma_{\xi+1})$, then $\mathrm{cf}(b(\beta)) \geq \mathrm{cf}(b(\gamma_\xi)) > \gamma_\xi \geq \xi$, whereas $f_{\alpha_\eta}(\beta) = g(\beta)$ for all $\eta > \xi$. □

Corollary 4.3.21

If κ is weakly compact, then D^b_κ does not add a dominating κ-real. ◁



A schematic representation of Figure 4.1, showing the effect of b'-κ-Hechler forcing with κ weakly compact.

Legend

- ◉ added by the forcing
- $*$ added for $b = b'$, unknown for other parameters
- • not added by the forcing
- ? unknown

Naturally we would like to iterate with D^b_κ and show that the iteration does not add dominating κ-reals either. There are several obstacles to this.

Firstly, we would need to show that not just the single step, but the whole iteration does not add dominating κ-reals. Secondly, we would need to preserve that κ is weakly compact along the iteration. Laver [Lav78] has given a construction of a forcing notion that makes a supercompact cardinal κ indestructible by $<\kappa$-directed closed forcing notions, but this argument is not directly of use to us, since D^b_κ is not $<\kappa$-directed closed (not even $<\kappa$-closed). We would therefore need to customise this argument for our purpose.

The above two issues are addressed by Shelah in the paper [She20].

κ-Localisation Forcing

Assumptions. We will assume that κ is inaccessible. We also need h to grow fast enough to avoid problems with Fodor's pressing down lemma. A sufficient assumption is that $h \in {}^\kappa\kappa$ is such that there exists a club set C with $h(\gamma) > \bigcup_{\alpha\in\gamma} h(\alpha)$ for all limit $\gamma \in C$, that is, h is discontinuous on C. One could compare this to the assumptions on h given in Theorem 3.4.5.

In a similar manner to how κ-Hechler forcing adds a dominating κ-real and a κ-Cohen generic, we can define κ-localisation forcing (with parameter h) as a forcing notion that adds an h-localising κ-real and a κ-Cohen generic. This forcing generalises the classical localisation forcing, and was used in its generalised form[10] in [BBTFM18, Section 4.3] to prove that $\mathfrak{b}^h_\kappa(\in^*)$ and $\mathfrak{d}^h_\kappa(\in^*)$ are consistently different from the bounds κ^+ and 2^κ.

As with κ-Hechler forcing, we will also consider a bounded variant of κ-localisation forcing in the next subsection, in order to study $\mathfrak{b}^{b,h}_\kappa(\in^*)$ and $\mathfrak{d}^{b,h}_\kappa(\in^*)$.

Definition 4.3.22
We define h-κ-*localisation forcing* $\mathbb{L}\mathrm{oc}^h_\kappa$ to have conditions (s,φ), where $s \in \mathrm{Loc}^h_{<\kappa}$ and $\varphi \in \mathrm{Loc}^h_\kappa$. The ordering is given by $(t,\psi) \le (s,\varphi)$ iff $s \subseteq t$ and $\varphi(\alpha) \subseteq \psi(\alpha)$ for all $\alpha \in \kappa \setminus \mathrm{dom}(t)$, and $\varphi(\alpha) \subseteq t(\alpha)$ for all $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$. ◁

[10] Our definition will differ slightly from the one in [BBTFM18], mainly to simplify one of our proofs in the next section on bounded κ-localisation forcing.

If $\mathbf{V} \subseteq \mathbf{W}$ are models of ZFC and there exists $\varphi \in (\mathrm{Loc}^h_\kappa)^{\mathbf{W}}$ such that $\{(s,\psi) \in (\mathbb{Loc}^h_\kappa)^{\mathbf{V}} \mid s \subseteq \varphi\}$ is a $(\mathbb{Loc}^h_\kappa)^{\mathbf{V}}$-generic filter over $\mathbf{V}$, then we call φ an ***h-κ-localisation generic (κ-real)***. For a condition (s,ψ), we can see s as an approximation of the h-κ-localisation generic φ, and ψ as a promise that $f \in^* \varphi$ holds for any f with $f \in^* \psi$. By the following this may include all f from the ground model:

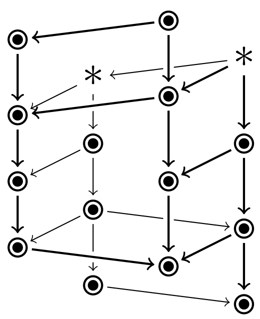
Lemma 4.3.23

$\mathbb{Loc}^h_\kappa$ adds an h-localising κ-real and a κ-Cohen generic. ◁

Proof. For any condition (s,ψ) and $f \in {}^\kappa\kappa$ in the ground model, also (s,ψ_f) is a condition, where $\psi_f : \alpha \mapsto \psi(\alpha) \cup \{f(\alpha)\}$ (remember that $h(\alpha)$ is an infinite cardinal). Clearly $(s,\psi_f) \leq (s,\psi)$, and if $\dot\varphi$ names the h-κ-localisation generic, then (s,ψ_f) “$f \in^* \dot\varphi$”. Hence $\dot\varphi$ names an h-localising κ-real.

The κ-Cohen generic is formed similarly to Lemma 4.3.12. □

Clearly an h'-localising κ-real φ is also h-localising if $h' \leq^* h$, but apart from this it is unknown whether $\mathrm{Loc}^{h'}_\kappa$ adds h-localising κ-reals for certain other parameters h.



A schematic representation of Figure 4.1, showing the effect of h'-κ-localisation forcing.

Legend
◉ added by the forcing
∗ added for $h \geq^* h'$, unknown for other parameters

Lemma 4.3.24

$\mathbb{Loc}^h_\kappa$ is strategically $<\kappa$-closed and $<\kappa^+$-c.c. ◁

Proof. White's winning strategy consists of choosing $(s_\alpha,\varphi_\alpha)$ such that $\alpha \leq \mathrm{dom}(s_\alpha) \in C$ and consequently $h(\mathrm{dom}(s_\alpha)) > |\bigcup_{\xi<\alpha}\varphi'_\xi(\beta)|$ for each $\beta \geq \mathrm{dom}(s_\alpha)$, where (s'_ξ,φ'_ξ) denote the previous Black moves.

As for $<\kappa^+$-c.c., it suffices to note that $\mathbb{Loc}^h_\kappa$ is $(\kappa,<\omega)$-centred. To clarify, $(s,\varphi) \parallel (s,\varphi')$ for all $s \in \mathrm{Loc}^h_{<\kappa}$, since (s,ψ) is below both, where $\psi(\alpha) = \varphi(\alpha) \cup \varphi'(\alpha)$ for all $\alpha \in \kappa \setminus \mathrm{dom}(s)$. □

Again, Lemma 4.1.20 applies, and thus the $<\kappa$-support iteration of $\mathbb{Loc}^h_\kappa$ is also $<\kappa^+$-c.c., and hence does not collapse cardinals or destroy the inaccessibility of κ.

We define the ***h-κ-localisation model*** as the result of forcing with a $<\kappa$-iteration of κ^{++} copies of $\mathbb{Loc}^h_\kappa$ over a ground model $\mathbf{V}$ “$2^\kappa = \kappa^+$”. We define the ***dual h-κ-localisation model*** as the result of forcing with a $<\kappa$-support iteration of κ^+ copies of $\mathbb{Loc}^h_\kappa$ over $\mathbf{V}$ “$2^\kappa = \kappa^{++}$”.

Theorem 4.3.25 — *[BBTFM18, Proposition 52]*

It is consistent that $\kappa^+ < \mathfrak{b}^h_\kappa(\in^*)$, since it holds in the h-κ-localisation model. ◁

Proof. The argument is as in Theorem 4.3.15, but now each stage adds an h-localising κ-real, which implies $\mathsf{b}^h_\kappa(\in^*) = \kappa^{++}$. □

Theorem 4.3.26 — *[BBTFM18, Proposition 52]*
It is consistent that $\mathsf{d}^h_\kappa(\in^*) = \kappa^+ < 2^\kappa$, since it holds in the dual h-κ-localisation model. ◁

Proof. As above, but here the κ^+ many h-localising κ-reals added in each stage form a witness for $\mathsf{d}^h_\kappa(\in^*) = \kappa^+$, while $2^\kappa > \kappa^+$ remains true, since no cardinals are collapsed. □

Bounded κ-Localisation Forcing

Assumptions. We assume that κ is inaccessible. We will also assume that $h \in {}^\kappa\kappa$ is such that case (iii) of Theorem 3.4.5 is satisfied, that is, h is discontinuous on the club set C.

Our main reason for introducing bounded κ-localisation forcing, is to show that case (iii) of Theorem 3.4.5 indeed leads to cardinal characteristics that are strictly different from κ^+ and 2^κ, in the same way we previously used bounded κ-Hechler forcing in Theorem 4.3.19. We also show that it is possible to add a (b, h)-localising κ-real without adding a dominating κ-real for weakly compact κ, strengthening what we saw for bounded κ-Hechler forcing.

Definition 4.3.27
We define (b, h)-κ-*localisation forcing* $\mathbb{L}\mathrm{oc}^{b,h}_\kappa$ to have conditions (s, φ) such that $s \in \mathrm{Loc}^{b,h}_{<\kappa}$ and $\varphi \in \mathrm{Loc}^{b,h}_\kappa$. The ordering is given by $(t, \psi) \leq (s, \varphi)$ iff $s \subseteq t$ and $\varphi(\alpha) \subseteq \psi(\alpha)$ for all $\alpha \in \kappa \setminus \mathrm{dom}(t)$, and $\varphi(\alpha) \subseteq t(\alpha)$ for all $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$. ◁

By the same logic as for (unbounded) κ-localisation forcing, $\mathbb{L}\mathrm{oc}^{b,h}_\kappa$ adds a (b, h)-localising κ-real and a κ-Cohen generic.

We can show that $\mathbb{L}\mathrm{oc}^{b,h}_\kappa$ is (κ, κ)-calibre when κ is weakly compact. We follow the proof of Lemma 4.3.20.

Lemma 4.3.28
Let $\langle \gamma_\xi \mid \xi \in \kappa \rangle$ enumerate the club set C. If κ is weakly compact and $\mathrm{cf}(h(\beta)) > \xi$ for each $\beta \in [\gamma_\xi, \gamma_{\xi+1})$, then $\mathbb{L}\mathrm{oc}^{b,h}_\kappa$ has (κ, κ)-calibre. ◁

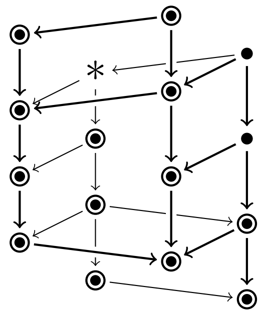
Proof. Let $s \in \mathrm{Loc}^{b,h}_{<\kappa}$ and $\{\varphi_\alpha \mid \alpha \in \kappa\} \subseteq \mathrm{Loc}^{b,h}_\kappa$, and assume without loss of generality that $\varphi_\alpha \neq \varphi_\beta$ and $s \subseteq \varphi_\alpha$ for all distinct $\alpha, \beta \in \kappa$

We define a κ-tree $T = \{t \in \mathrm{Loc}^{b,h}_{<\kappa} \mid \exists\alpha\exists\beta(\alpha \neq \beta \wedge t \subseteq \varphi_\alpha \cap \varphi_\beta)\}$. Since κ is weakly compact, there exists a branch $\psi \in [T]$ and for any $\alpha_0, \gamma \in \kappa$ there exists some $\alpha \geq \alpha_0$ such that $\gamma \subseteq \mathrm{dom}(\varphi_\alpha \cap \psi)$.

For each ξ find some α_ξ with $\gamma_\xi + 1 \subseteq \mathrm{dom}(\varphi_{\alpha_\xi} \cap \psi)$ such that $\langle \alpha_\xi \mid \xi \in \kappa \rangle$ is strictly increasing. Let $A = \{\alpha_\xi \mid \xi \in \kappa\}$ and $\varphi : \beta \mapsto \bigcup_{\xi\in\kappa} \varphi_{\alpha_\xi}(\beta)$, then these are as needed. Note that $\varphi(\beta) = \bigcup_{\eta\in\xi} \varphi_{\alpha_\eta}(\beta)$ if $\beta \in [\gamma_\xi, \gamma_{\xi+1})$, and thus $|\varphi(\beta)| < h(\beta)$, because we assumed $\mathrm{cf}(h(\beta)) > \xi$. □

Corollary 4.3.29
Under the assumptions of the lemma, $\mathbb{L}\mathrm{oc}^{b,h}_\kappa$ does not add a dominating κ-real. ◁

As with (unbounded) κ-localisation forcing, it is not clear whether $\mathrm{Loc}_\kappa^{b,h}$ adds (b',h')-localising κ-reals for certain other parameters b', h'.



A schematic representation of Figure 4.1, showing the effect of (b,h')-κ-localisation forcing under the assumptions of Lemma 4.3.28.

Legend
- ◉ added by the forcing
- • not added by the forcing
- $\ast$ added for $h = h'$, unknown for other parameters

Lemma 4.3.30
$\mathsf{Loc}_\kappa^{b,h}$ is strategically $<\kappa$-closed and $<\kappa^+$-c.c. ◁

Proof. See the proof of Lemma 4.3.24. □

The above lemma also shows, together with Lemma 4.1.20, that $<\kappa$-iteration of $\mathsf{Loc}_\kappa^{b,h}$ is $<\kappa^+$-c.c., and hence preserves cardinals.

Theorem 4.3.31
It is consistent that $\mathfrak{b}_\kappa^{b,h}(\in^*) > \kappa^+$ and that $\mathfrak{d}_\kappa^{b,h}(\in^*) < 2^\kappa$. ◁

Proof. Analogous to Theorem 4.3.19, with $\mathsf{Loc}_\kappa^{b,h}$ taking the role of D_κ^b. □

κ-Eventually Different Forcing

Assumptions. We assume that κ is regular uncountable.

Bounded κ-Hechler and bounded κ-localisation forcing notions showed that we can add an eventually different κ-real without adding a dominating κ-real, although this required κ to be weakly compact. The question is whether we can do better, and add an eventually different κ-real without adding dominating κ-reals, but also no b-dominating κ-real, or even b-eventually different κ-real.

The forcing notion we use for this is a generalisation of the classical eventually different forcing of Miller [Mil81, Section 5]. This forcing notion does not add dominating reals, proved using a compactness argument, and is furthermore σ-centred. We will show that the higher version is $(\kappa, <\kappa)$-centred (with canonical bounds) and does not add a dominating κ-real. For the latter property, we need to assume a certain amount of compactness for κ. In particular, it suffices that κ is weakly compact.

Definition 4.3.32
We define *κ-eventually di erent forcing* E_κ to have conditions (s,F) such that $s \in {}^{<\kappa}\kappa$ and $F \in [{}^\kappa\kappa]^{<\kappa}$, where the ordering is defined by $(t,G) \leq (s,F)$ iff $s \subseteq t$ and $F \subseteq G$ and for $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ we have $t(\alpha) \notin \{f(\alpha) \mid f \in F\}$. ◁

If $\mathbf{V} \subseteq \mathbf{W}$ are models of ZFC and there exists $f \in ({}^\kappa\kappa)^{\mathbf{W}}$ such that $\{(s, F) \in (\mathsf{E}_\kappa)^{\mathbf{V}} \mid s \subseteq f\}$ is an $(\mathsf{E}_\kappa)^{\mathbf{V}}$-generic filter over $\mathbf{V}$, then we call f a *κ-eventually di erent generic (κ-real)*.[11]

Lemma 4.3.33

E_κ is $<\kappa$-closed and $(\kappa, <\kappa)$-centred. ◁

Proof. For closure, let $\langle (s_\alpha, F_\alpha) \mid \alpha \in \lambda \rangle$ be a decreasing sequence of conditions in E_κ of length $\lambda < \kappa$, then $s = \bigcup_{\alpha\in\lambda} s_\alpha \in {}^{<\kappa}\kappa$ by compatibility of the conditions and regularity of κ. Let $F = \bigcup_{\alpha\in\lambda} F_\alpha$, then $|F| < \kappa$. Clearly $(s, F) \leq (s_\alpha, F_\alpha)$ for all $\alpha \in \kappa$.

For centredness, define for each $s \in {}^{<\kappa}\kappa$ the subset $E_s = \{s\} \times [{}^\kappa\kappa]^{<\kappa} \subseteq \mathsf{E}_\kappa$. If $X \subseteq E_s$ and $|X| < \kappa$, then let $F = \bigcup_{(s,F')\in X} F'$, then $F \in [{}^\kappa\kappa]^{<\kappa}$. Hence $(s, F) \leq (s, F')$ for all $(s, F') \in X$, showing that E_s is $<\kappa$-linked. □

Lemma 4.3.34

E_κ does not add a b-eventually different κ-real. ◁

Proof. By Lemma 4.2.3 □

In order to discuss the relation between E_κ and dominating κ-reals, we will need a generalised form of compactness and of Tychonoff's theorem where "finite" is replaced by "$<\kappa$".

Definition 4.3.35

The weight of a topological space (X, τ) is the least cardinality of a base $\mathcal{B}$ for τ. We say X is $<\kappa$-compact if for every $C \subseteq \tau$ such that $\bigcup C = X$ there exists $C' \in [C]^{<\kappa}$ such that $\bigcup C' = X$.

A cardinal κ is called ***weakly square compact*** if for any $<\kappa$-compact topological space (X, τ) with $w(X) \leq \kappa$ also $X \times X$ with the product topology is $<\kappa$-compact. ◁

If we assume that κ is strongly compact[12], then the generalised form of Tychonoff's theorem holds, in the sense that the $<\kappa$-box product (see the definition below) of $<\kappa$-compact spaces is $<\kappa$-compact. However, in our specific case it suffices to assume that κ is weakly compact.

Definition 4.3.36

Let I be a set of indices and X_i a topological space for each $i \in I$ and remember that the $<\kappa$-box topology on the product $X = \prod_{i\in I} X_i$ is defined as the topology generated by basic open sets of the form $[s] = \{f \in X \mid s \subseteq f\}$ for $s \in \prod_{i\in I'} X_i$ with $I' \in [I]^{<\kappa}$. We call X the *$<\kappa$-box product* of $\{X_i \mid i \in I\}$. ◁

Theorem 4.3.37 — *[BD21, Theorem 5.1]*

The following are equivalent:

- κ is weakly compact.

[11] Note that an *eventually di erent κ-real* is not the same as a *κ-eventually di erent generic*. We use the former to describe κ-reals with combinatorial properties over the ground model, as in Definition 4.2.1, and the latter to describe the specific generic κ-real added by E_κ.

[12] See [BD21, Theorem 2.10]

- κ is weakly square compact.
- if $\{X_i \mid i \in I\}$ is a family of spaces where $|I| \leq \kappa$ and each X_i is a $<\kappa$-compact space with $w(X_i) \leq \kappa$, then the $<\kappa$-box product of $\{X_i \mid i \in I\}$ is also $<\kappa$-compact.

To show that E_κ does not add dominating κ-reals, we now follow the proof of Miller. We first need a preliminary lemma.

Lemma 4.3.38 — *cf. [Mil81, Lemma 5.1] for ${}^\omega\omega$*
Let $\dot{x}$ be an E_κ-name for a set in $\mathbf{V}$, let $s \in {}^{<\kappa}\kappa$ and $\lambda \in \kappa$, then there exists a set X with $|X| < \kappa$ such that for all $F \in [{}^\kappa\kappa]^\lambda$ there exists $p \leq (s, F)$ such that p "$\dot{x} \in X$". ◁

Proof. Let κ have the cobounded topology τ given by $X \in \tau$ if and only if $[\alpha, \kappa) \subseteq X$ for some $\alpha \in \kappa$. It is easily seen that κ is $<\kappa$-compact and $w(\kappa) = \kappa$, since $|\tau| = \kappa$. Therefore, if we give ${}^\kappa\kappa$ the $<\kappa$-box topology where κ has the cobounded topology, then ${}^\kappa\kappa$ is $<\kappa$-compact by Theorem 4.3.37, and more generally we see that ${}^\lambda({}^\kappa\kappa)$ is $<\kappa$-compact for any cardinal $\lambda < \kappa$. When referring to ${}^\lambda({}^\kappa\kappa)$ or ${}^\kappa\kappa$ within the proof of this claim, it will be with respect to this topology.

Fix $\dot{x}$, s and λ as in the claim. Given a sequence $F \in {}^\lambda({}^\kappa\kappa)$, let $\tilde{F} = \mathrm{ran}(F) \in [{}^\kappa\kappa]^\lambda$. Given some X with $|X| < \kappa$ and t with $s \subseteq t$, we define the following two sets:

$$\mathcal{F}_X = \left\{F \in {}^\lambda({}^\kappa\kappa) \;\middle|\; \exists p \in \mathsf{E}_\kappa(p \leq (s, \tilde{F}) \text{ and } p \quad \text{“}\dot{x} \in X\text{”})\right\},$$
$$\mathcal{U}_t = \left\{F \in {}^\lambda({}^\kappa\kappa) \;\middle|\; \forall \xi \in \mathrm{dom}(t) \setminus \mathrm{dom}(s) \forall \eta \in \lambda(t(\xi) \neq F(\eta)(\xi))\right\}.$$

Note that for any $F \in {}^\lambda({}^\kappa\kappa)$ there is $y \in \mathbf{V}$ with $F \in \mathcal{F}_{\{y\}}$, since $(s, \tilde{F})$ "$\exists y \in \mathbf{V}(\dot{x} = y)$", hence there exists $p \leq (s, \tilde{F})$ and $y \in \mathbf{V}$ such that p "$\dot{x} = y$". Also note that $\mathcal{U}_t$ is open in the topology on ${}^\lambda({}^\kappa\kappa)$.

We show that $\mathcal{F}_X$ is open as well. For each $F \in \mathcal{F}_X$, choose some $(t, H) \leq (s, \tilde{F})$ such that (t, H) "$\dot{x} \in X$", then by definition of the order on E_κ we see that $F \in \mathcal{U}_t$. Furthermore, let $K \in \mathcal{U}_t$, then we have $(t, H \cup \tilde{K}) \leq (s, \tilde{F} \cup \tilde{K}) \leq (s, \tilde{K})$ and $(t, H \cup \tilde{K})$ "$\dot{x} \in X$" because $(t, H \cup \tilde{K}) \leq (t, H)$, hence $K \in \mathcal{F}_X$ and thereby $\mathcal{U}_t \subseteq \mathcal{F}_X$. In conclusion, $\mathcal{F}_X$ is the union of sets of the form $\mathcal{U}_t$, and hence is open.

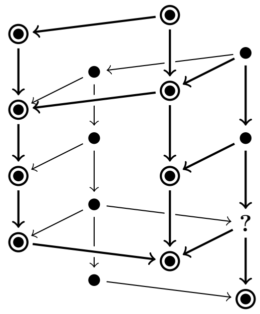
Combined, we see that there is a set $\mathcal{Y}$ of size $2^\kappa = |{}^\lambda({}^\kappa\kappa)|$ with ${}^\lambda({}^\kappa\kappa) = \bigcup_{y \in \mathcal{Y}} \mathcal{F}_{\{y\}}$. We can use $<\kappa$-compactness to find a family $\mathcal{X} \subseteq \mathcal{Y}$ with $|\mathcal{X}| < \kappa$ such that $\bigcup_{y \in \mathcal{X}} \mathcal{F}_{\{y\}} = {}^\lambda({}^\kappa\kappa)$. Then note that $\mathcal{F}_{X \cup X'} \supseteq \mathcal{F}_X \cup \mathcal{F}_{X'}$ to conclude that $\mathcal{F}_\mathcal{X} = {}^\lambda({}^\kappa\kappa)$. Therefore $\mathcal{X}$ is the set we are looking for to complete the proof of the claim. □

Theorem 4.3.39 — *cf. [Mil81, Section 5] for ${}^\omega\omega$*
If κ is weakly compact, then E_κ does not add dominating κ-reals. ◁

Proof. Let $\langle (s_\eta, \lambda_\eta) \mid \eta \in \kappa \rangle$ list all (s, λ) with $s \in {}^{<\kappa}\kappa$ and $\lambda < \kappa$ a cardinal such that each $(s, \lambda) = (s_\eta, \lambda_\eta)$ for κ many $\eta \in \kappa$.

Let $\dot{f}$ be an E_κ-name and $\Vdash_{\mathsf{E}_\kappa}$ "$\dot{f} \in {}^\kappa\kappa$". Given $\eta \in \kappa$, use the claim to find a set $X_\eta \in [\kappa]^{<\kappa}$ such that for all $F \in [{}^\kappa\kappa]^{\lambda_\eta}$ there exists $p \leq (s_\eta, F)$ such that $p \Vdash$ "$\dot{f}(\eta) \in X_\eta$". Then we let $g : \eta \mapsto \sup(X_\eta) + 1$ for each $\eta \in \kappa$.

Let $(s, F) \in \mathsf{E}_\kappa$ and $\eta_0 \in \kappa$, then there exists $\eta \geq \eta_0$ such that $(s, |F|) = (s_\eta, \lambda_\eta)$. By the claim there exists $p \leq (s, F)$ such that $p \Vdash$ "$\dot{f}(\eta) \in X_\eta$" and thus $p \Vdash$ "$\dot{f}(\eta) < g(\eta)$". Since (s, F) and η_0 are arbitrary, we see that $\Vdash_{\mathsf{E}_\kappa}$ "$g \leq^* f$", thus $\dot{f}$ does not name a dominating κ-real. $\square$



A schematic representation of Figure 4.1, showing the effect of κ-eventually different forcing with κ weakly compact.

Legend
- ◉ added by the forcing
- • not added by the forcing
- ? unknown

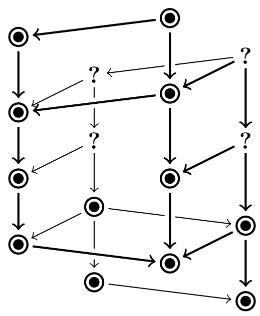
Bounded κ-Antiavoidance Forcing

Assumptions. We assume κ is inaccessible. Furthermore, we need to assume that the conditions of case (iii) of Theorem 3.4.16 are satisfied, hence we let C be a club set such that b is discontinuous on C and $\mathrm{cf}(b)$ is increasing and discontinuous on $\{\alpha \in C \mid h(\alpha) = b(\alpha)\}$.

In order to show that $\mathfrak{b}^{b,h}_\kappa(\ni^\infty)$ and $\mathfrak{d}^{b,h}_\kappa(\ni^\infty)$ are consistently different from κ^+ and 2^κ when the assumptions of case (iii) of Theorem 3.4.16 are satisfied, we introduce another bounded forcing notion based on antilocalisation and antiavoidance. Note that we do not need an unbounded form of this forcing to show the same for $\mathfrak{b}^h_\kappa(\ni^\infty)$ and $\mathfrak{d}^h_\kappa(\ni^\infty)$, since these are equal to $\mathrm{non}(\mathcal{M}_\kappa)$ and $\mathrm{cov}(\mathcal{M}_\kappa)$ by Corollary 3.3.8.

Definition 4.3.40
We define (b, h)-κ-*antiavoidance forcing* $\mathsf{AA}\mathbb{v}^{b,h}_\kappa$ to have conditions (s, Φ) where $s \in \prod_{<\kappa} b$ and $\Phi \subseteq \mathrm{Loc}^{b,h}_\kappa$ with $|\Phi| \leq \mathrm{dom}(s)$. The ordering is given by $(t, \Psi) \leq (s, \Phi)$ if $s \subseteq t$ and $\Phi \subseteq \Psi$ and for each $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ and $\varphi \in \Phi$ we have $t(\alpha) \notin \varphi(\alpha)$. $\triangleleft$

It is easy to see that $\mathsf{AA}\mathbb{v}^{b,h}_\kappa$ adds a (b, h)-antiavoiding κ-real and a κ-Cohen generic.



A schematic representation of Figure 4.1, showing the effect of (b, h)-κ-antiavoidance forcing

Legend
- ◉ added by the forcing
- • not added by the forcing
- ? unknown

Lemma 4.3.41
$\mathsf{AA}\mathbb{v}_\kappa^{b,h}$ is strategically $<\kappa$-closed and $<\kappa^+$-c.c. ◁

Proof. For strategic $<\kappa$-closure, at stage α of $\mathcal{G}(\mathsf{AA}\mathbb{v}_\kappa^{b,h}, p)$, let $\langle p_\xi = (s_\xi, \Phi_\xi) \mid \xi \in \alpha\rangle$ and $\langle p'_\xi = (s'_\xi, \Phi'_\xi) \mid \xi \in \alpha\rangle$ be the sequences of previous moves by White and Black respectively. We will describe the winning strategy for White at stage α.

If $\alpha = \beta + 1$ is successor, White will decide some $\gamma \in C$ such that $\alpha \leq \gamma$. Let $\mathrm{dom}(s'_\beta) = \gamma' \in C$, then if $\xi \in [\gamma', \gamma)$ we have $|\bigcup_{\varphi \in \Phi'_\beta} \varphi(\xi)| \leq |\gamma'| \cdot \sup_{\varphi\in\Phi'_\beta} |\varphi(\xi)| < b(\xi)$ through the same reasoning as in the proof of Theorem 3.4.16 (iii). Therefore, we can find some value $s_\alpha(\xi) \in b(\xi) \setminus \bigcup_{\varphi\in\Phi'_\beta} \varphi(\xi)$ to define s_α with $\mathrm{dom}(s_\alpha) = \gamma$, and let $\Phi_\alpha = \Phi'_\beta$.

We have to show that White can make a move at limit α as well. We claim that $s_\alpha = \bigcup_{\xi\in\alpha} s_\xi$ and $\Phi_\alpha = \bigcup_{\xi\in\alpha} \Phi_\xi$ works. Clearly $\mathrm{dom}(s_\alpha) \in C$ by C being club, and

$$|\Phi_\alpha| \leq |\alpha| \cdot \sup_{\xi\in\alpha} |\Phi_\xi| = |\alpha| \cdot \bigcup_{\xi\in\alpha} \mathrm{dom}(s_\xi) \leq |\alpha| \leq \mathrm{dom}(s_\alpha).$$

Therefore, (s_α, F_α) is indeed a valid move for White, so White has a winning strategy.

For $<\kappa^+$-c.c., note that for any $(s, \Phi), (s, \Psi) \in \mathsf{AA}\mathbb{v}_\kappa^{b,h}$ with $\mathrm{dom}(s) = \gamma$ we have $|\Phi| \leq \gamma$ and $|\Psi| \leq \gamma$. We may assume without loss of generality that γ is infinite, then also $|\Phi \cup \Psi| \leq \gamma$, hence $(s, \Phi \cup \Psi)$ is a condition below both (s, Φ) and (s, Ψ). Thus if $\mathcal{A} \subseteq \mathsf{AA}\mathbb{v}_\kappa^{b,h}$ is an antichain and $(s, \Phi), (t, \Psi) \in \mathcal{A}$ are distinct, then we must have $s \neq t$, hence $|\mathcal{A}| \leq \kappa$. □

Corollary 4.3.42
$\mathsf{AA}\mathbb{v}_\kappa^{b,h}$ preserves cardinals and cofinality and does not add any elements to ${}^{<\kappa}\kappa$. ◁

Theorem 4.3.43
It is consistent that $\mathsf{b}_\kappa^{b,h}(\ni^\infty) > \kappa^+$ and that $\mathsf{d}_\kappa^{b,h}(\ni^\infty) < 2^\kappa$. ◁

Proof. Analogous to Theorem 4.3.19, with $\mathsf{AA}\mathbb{v}_\kappa^{b,h}$ taking the role of D_κ^b. □

4.4. Forcing Notions and Perfect Trees

So far, we have seen several forcing notions that add κ-Cohen generics. We will now introduce some forcing notions that do not add κ-Cohen reals. We will describe each of the forcing notions in terms of perfect trees firstly, and then study their properties in the subsections.

For the entirety of this section we will assume that κ is inaccessible.

Definition 4.4.1
Let $\mathcal{F}$ be a set of functions such that $\mathrm{dom}(f) = \kappa$ for each $f \in \mathcal{F}$, where we usually assume $\mathcal{F} = {}^\kappa\kappa$ or $\mathcal{F} = {}^\kappa 2$. Given a κ-tree $T \subseteq \mathcal{F}_{<\kappa}$, we define the following properties:

- T is *perfect* if for every $u \in T$ there exists $v \supseteq u$ such that $v \in \mathrm{Split}(T)$.
- T is *Laver* if there exists $u \in T$ such that for all $v \in T$ we have $v \supseteq u$ iff $v \in \mathrm{Split}(T)$.
- T is *closed (under splitting)* if $\bigcup C \in \mathrm{Split}(T)$ for every chain $C \subseteq \mathrm{Split}(T)$ with $|C| < \kappa$.

- Let $\mathcal{U}$ be a family of sets (e.g. a filter on κ), then T is ***guided by*** $\mathcal{U}$ if $\{x \mid u^\frown\langle x\rangle \in T\} \in \mathcal{U}$ for all $u \in \operatorname{Split}(T)$.
- T is ***uniform*** if for every $\alpha \in \kappa$ there is a set X such that $\operatorname{suc}(u, T) = \{u^\frown\langle x\rangle \mid x \in X\}$ for all $u \in \operatorname{Lev}_\alpha(T)$.

Let $\mathcal{U}$ be a $<\kappa$-complete nonprincipal filter on κ. We then define the following forcing notions:

- *κ-Sacks forcing* S_κ has closed perfect trees $T \subseteq {}^{<\kappa}2$ as conditions.
- *κ-Silver forcing* V_κ has closed uniform perfect trees $T \subseteq {}^{<\kappa}2$ as conditions.
- *κ-Miller forcing* $\mathsf{Mi}^{\mathcal{U}}_\kappa$ has closed perfect trees $T \subseteq {}^{<\kappa}\kappa$ guided by $\mathcal{U}$ as conditions.
- *κ-Laver forcing* $\mathsf{L}^{\mathcal{U}}_\kappa$ has closed Laver trees $T \subseteq {}^{<\kappa}\kappa$ guided by $\mathcal{U}$ as conditions.

Each of these forcing notions is ordered by $T \leq S$ iff $T \subseteq S$.[13] ◁

Note that Laver trees are perfect. Also note that if T is a closed perfect tree, then $\operatorname{Lev}_\alpha(T) \neq$? for all $\alpha \in \kappa$ and for any $u \in T$ with $\operatorname{dom}(u) \leq \alpha$ there exists $v \in \operatorname{Split}_\alpha(T)$ such that $u \subseteq v$.

When mentioning $\mathcal{U}$ in the remainder of this chapter, we will assume that $\mathcal{U}$ is a $<\kappa$-complete nonprincipal filter.

Lemma 4.4.2
If $T \subseteq \mathcal{F}_{<\kappa}$ is a closed perfect tree and $b \in [T]$ is a branch of T, then $\operatorname{dom}(b) = \kappa$. ◁

Proof. Suppose that $C \subseteq T$ is a chain such that $u = \bigcup C$ and $\operatorname{dom}(u) \in \kappa$. We will show that $u \in T$, and hence that T contains some $v \supseteq u$ that is splitting (because T is perfect). This implies that C is not a branch.

Let $C' = \{v \subseteq u \mid v \in \operatorname{Split}(T)\}$. Either $\bigcup C' = u$, in which case follows that $u \in T$ by closure under splitting, or $\bigcup C' = v$ (u, in which case $w = u$ $(\operatorname{dom}(v) + 1) \in \operatorname{suc}(v, T) \subseteq T$, and consequently by T being perfect and the fact that no $w' \in T$ with $w \subseteq w' \subseteq u$ is splitting there exists some $u' \supseteq u$ such that $u' \in \operatorname{Split}(T)$, which implies $u \in T$ as well. □

The following lemma shows that the properties mentioned in Definition 4.4.1 are preserved by certain subtrees. We will omit the proof, as it is quite evident.

Lemma 4.4.3
Let T be a κ-tree and $u \in T$. If T has property $P \in \{$"perfect", "Laver", "closed under splitting", "guided by $\mathcal{U}$", "uniform"$\}$, then $(T)_u$ also has property P.

Finally each of these forcing notions has a fusion ordering and is closed under fusion. We will prove this for each of the above forcing notions simultaneously, by going through the properties one by one. Fusion would not be powerful if the forcing notion is not also $<\kappa$-closed, which allows us to actually construct fusion sequences. We therefore state the following lemma in this section as well, but omit the proof, firstly because it can be found elsewhere and secondly because we will prove a $<\kappa$-closure for the forcing of Chapter 5 in Lemma 5.1.2, which is similar enough to apply to the forcing notions of this section.

[13] Note that we can describe κ-Cohen forcing as the set of closed Laver trees guided by $U = fXg$, where X is any set with 2 jXj κ.

Lemma 4.4.4 — *Folklore, cf. Lemma 5.1.2*
Let $\mathsf{P} \in \{\mathsf{S}_\kappa, \mathsf{V}_\kappa, \mathsf{Mi}^{\mathcal{U}}_\kappa\}$, then P is $<\kappa$-closed. ◁

Remember that $\mathcal{U}$ is a $<\kappa$-complete filter, which is necessary for the $<\kappa$-closure of $\mathsf{Mi}^{\mathcal{U}}_\kappa$. We will define an ordering that we may use as fusion ordering. In the case of $\mathsf{Mi}^{\mathcal{U}}_\kappa$ where $\mathcal{U}$ is a $<\kappa$-complete *normal* filter, we can describe a different, more versatile, fusion ordering that resembles the usual fusion ordering on (classical) Laver and Miller forcing, see for example [FZ10, Definition 2.2]. The following simpler fusion ordering will be sufficient for our needs, however.

Definition 4.4.5
Let $\mathsf{P} \in \{\mathsf{S}_\kappa, \mathsf{V}_\kappa, \mathsf{Mi}^{\mathcal{U}}_\kappa\}$. For $T, S \in \mathsf{P}$, we let $T \leq_\alpha S$ iff $T \leq S$ and $\mathrm{Split}_\alpha(T) = \mathrm{Split}_\alpha(S)$. ◁

Lemma 4.4.6 — *Folklore*
The orders $\langle \leq_\alpha \mid \alpha \in \kappa \rangle$ in Definition 4.4.5 are a fusion ordering (see Section 1.2) and P is closed under fusion. ◁

Proof. Let $\langle T_\alpha \mid \alpha \in \kappa \rangle$ be a fusion sequence, that is, $T_\beta \leq_\alpha T_\alpha$ for all $\alpha \leq \beta \in \kappa$. We claim that $T = \bigcap_{\alpha \in \kappa} T_\alpha \in \mathsf{P}$. It is not hard to see that T is a κ-tree and $T \leq_\alpha T_\alpha$ for each $\alpha \in \kappa$.

Closed under splitting. Let $C \subseteq T$ be a chain of splitting nodes in T and $|C| < \kappa$, then by regularity of κ there is $\alpha \in \kappa$ such that each $u \in C$ is in $\mathrm{Split}_\xi(T)$ for some $\xi < \alpha$. But $\mathrm{Split}_\xi(T) = \mathrm{Split}_\xi(T_\beta)$ for all $\beta \geq \xi$, hence C is a chain of splitting nodes in T_α. Then $\bigcup C \in T_\alpha$ is a splitting node, say $\bigcup C \in \mathrm{Split}_\gamma(T_\alpha)$. By choice of α, we have $\gamma \leq \alpha$, and thus $\bigcup C \in \mathrm{Split}_\gamma(T_\beta)$ for all $\beta \geq \alpha$. Hence $\bigcup C \in \mathrm{Split}_\gamma(T)$.

Perfect. Let $u \in T$ and $\mathrm{dom}(u) = \alpha$, then $u \in T_{\alpha+1}$ and there exists $v \in \mathrm{Split}_\alpha(T_{\alpha+1})$ such that $u \subseteq v$, since T_α is a closed perfect tree. But $\mathrm{Split}_{\alpha+1}(T_{\alpha+1}) = \mathrm{Split}_{\alpha+1}(T_\beta)$ for all $\beta \geq \alpha + 1$, hence $\mathrm{suc}(v, T_{\alpha+1}) = \mathrm{suc}(v, T_\beta)$ for all $\beta \geq \alpha + 1$ and consequently $v \in T$ and $\mathrm{suc}(v, T_{\alpha+1}) = \mathrm{suc}(v, T)$, making $v \supseteq u$ a splitting node in T.

Guided by $\mathcal{U}$. Suppose each T_α is guided by $\mathcal{U}$ and let $u \in T$ be a splitting node, then $u \in \mathrm{Split}_\alpha(T_{\alpha+1})$ for some $\alpha \in \kappa$, and thus $\{x \mid u^\frown\langle x \rangle \in T\} = \{x \mid u^\frown\langle x \rangle \in T_\beta\}$ for each $\beta \geq \alpha + 1$ by the same argument as in the "perfect" case above, hence the set of values extending u in T form a set in $\mathcal{U}$.

Uniform. Suppose each T_α is uniform, $\alpha \in \kappa$ and $u \in \mathrm{Lev}_\alpha(T)$. Note that $\mathrm{suc}(u, T) = \mathrm{suc}(u, T_\beta)$ for all $\beta \geq \alpha + 1$ and each such T_β is uniform, therefore T is uniform. □

κ-Sacks and κ-Silver Forcing

The first forcing notions we discuss are κ-Sacks forcing S_κ and κ-Silver forcing V_κ. Kanamori [Kan80] has studied κ-Sacks forcing first for general regular uncountable κ, although it behaves quite differently from classical Sacks forcing if κ is accessible.

In the context of ${}^\omega\omega$, many cardinal characteristics known to be *tame* remain small in the Sacks model[14] and it is therefore a natural candidate to consider if one wishes to show the consistency of $\mathfrak{x} < 2^{\aleph_0}$ for some cardinal characteristic $\mathfrak{x}$.

[14] See [Zap08, Chapter 6].

In the context of ${}^\kappa\kappa$, κ-Sacks forcing is more subtle, and by [BBTFM18, Theorem 70] it can be used to prove the consistency of $\mathsf{d}_\kappa^{\mathsf{pow}}(\in^*) < \mathsf{d}_\kappa^{\mathsf{id}}(\in^*)$.[15] To show this, one first notes that S_κ has the pow-Sacks property, but not the id-Sacks property, and thus adds an id-avoiding κ-real and no pow-avoiding κ-reals by Lemma 4.2.10. Alternatively, one can prove the same things for κ-Silver forcing, because the uniformity of the conditions does not affect the proof.

Lemma 4.4.7 — *[BBTFM18, Lemma 69]*
S_κ and V_κ have the pow-Sacks property, but not the id-Sacks property. ◁

We will not repeat a proof at this moment, but in Chapter 5 we will provide proof of more general statements in order to separate many different cardinals of the form $\mathsf{d}_\kappa^h(\in^*)$ as Theorems 5.1.5 and 5.1.8.

The other important step is that $\leq\kappa$-support products also have the pow-Sacks property. Alternatively, one can also use $\leq\kappa$-support iteration, instead of $\leq\kappa$-support products, which also preserves the pow-Sacks property. We will prove the more general preservation results in Lemma 5.2.5.

The pow-Sacks property implies that S_κ and V_κ do not add pow-avoiding κ-reals, hence also no unbounded κ-reals. Moreover, if $\mathsf{pow} < \mathrm{cf}(b)$, that is, if $\mathsf{pow}(\alpha) = (2^{|\alpha|})^+ < \mathrm{cf}(b(\alpha))$ for all α, then it is clear that S_κ and V_κ have the (b, pow)-Laver property, and thus do not add (b, pow)-avoiding κ-reals either. For any such b, no b-unbounded κ-reals are added either.

On the other hand, we know that an id-avoiding κ-real is added, and in fact with the same methods we can show that a (b, id)-avoiding κ-real is added as well. This construction is done using fusion, and we will come back to this in the next chapter. In this chapter we will only use fusion to show that S_κ does not add eventually different κ-reals. A similar proof can be done for V_κ, but as the argument is complicated by the uniformity of the conditions, we will not do so.

Theorem 4.4.8
S_κ does not add eventually different κ-reals. ◁

Proof. Let $\dot f$ be a name and $T_0 \in \mathsf{S}_\kappa$ be such that $T_0 \Vdash$ "$\dot f \in {}^\kappa\kappa$", then we will use fusion to construct a condition $T \leq T_0$ and a κ-real $g \in {}^\kappa\kappa$ such that $T \Vdash$ "$\dot f =^\infty \check g$". This implies that $\dot f$ is not eventually different over $\mathbf{V}$.

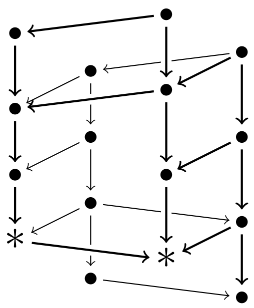
Suppose T_α has been defined and that we have defined g up to δ_α, that is, $g \restriction \delta_\alpha$ is defined and $g(\delta_\alpha)$ not yet. Let $\langle v_\xi \mid \xi \in \lambda\rangle$ be an enumeration of all $v \in T_\alpha$ such that $v = u^\frown\langle x\rangle$ for some $u \in \mathrm{Split}_\alpha(T_\alpha)$, and note that $\lambda < \kappa$, because κ is inaccessible and there is an obvious order-preserving bijection between $\langle {}^{\leq\alpha}2, \subseteq\rangle$ and $\langle \mathrm{Split}_{\leq\alpha}(T_\alpha), \subseteq\rangle$.

For each $\xi \in \lambda$, find some $T_\alpha^\xi \leq (T_\alpha)_{v_\xi}$ such that there exists $\beta_\xi \in \kappa$ with $T_\alpha^\xi \Vdash$ "$\dot f(\delta_\alpha + \xi) = \check\beta_\xi$", and define $g(\delta_\alpha + \xi) = \beta_\xi$. Then we define $T_{\alpha+1} = \bigcup T_\alpha^\xi$. It is not difficult to see that $T_{\alpha+1} \in \mathsf{S}_\kappa$ and $T_{\alpha+1} \leq_\alpha T_\alpha$.

[15] Remember that $\mathsf{id} : \alpha \mapsto |\alpha|^+$ and $\mathsf{pow} : \alpha \mapsto (2^{|\alpha|})^+$.

Finally, if $\alpha \in \kappa$ is limit, we define $T_\alpha = \bigcap_{\xi<\alpha} T_\xi$, then $T_\alpha \in \mathsf{S}_\kappa$ by Lemma 4.4.4, and it is easy to check that $T_\alpha \leq_\xi T_\xi$ for each $\xi < \alpha$.

Finally we let T be the fusion limit of $\langle T_\alpha \mid \alpha \in \kappa\rangle$, then T “$\dot f =^\infty \breve g$”: if G is a generic filter with $T \in G$, then we can find some $u \in \mathrm{Split}_\alpha(T)$ and $v \in \mathrm{suc}(u, T)$ for any $\alpha \in \kappa$ such that $(T)_v \in G$, and clearly $(T)_v$ “$\dot f(\delta_\alpha + \xi) = g(\delta_\alpha + \xi)$” for ξ such that $v = v_\xi$ by construction. □



A schematic representation of Figure 4.1, showing the effect of κ-Sacks / κ-Silver forcing with $\mathrm{cf}(b) >$ pow.

Legend

- $\bullet$ not added by the forcing
- $*$ added for $h = \mathsf{id}$, but not for $h \geq^* \mathsf{pow}$

We will conclude this section with an independence result concerning $\mathcal{SN}_\kappa$, which shows that Theorem 3.3.16 cannot be dualised. The proof is based on [GJS93], where a more complicated[16] perfect tree forcing is used.

Theorem 4.4.9 — *cf. [GJS93] for* ${}^\omega\omega$

Let $\bar{\mathsf{S}}$ be the $\leq\kappa$-support iteration of S_κ of length κ^{++}. If $\mathbf{V}$ “$2^\kappa = \kappa^+$”, then

$$\mathbf{V}^{\bar{\mathsf{S}}} \text{ “}\mathrm{cof}(\mathcal{M}_\kappa) = \kappa^+ < \kappa^{++} = \mathrm{cov}(\mathcal{SN}_\kappa)\text{”}. \qquad \triangleleft$$

Proof. S_κ is ${}^\kappa\kappa$-bounding, since it has the pow-Sacks property. Because of this, there exists a $\subseteq$-cofinal subset of $\mathcal{SN}_\kappa$ whose elements can be coded terms of a family of sequences indexed by some fixed dominating family. Since the dominating family remains dominating, we can essentially characterise the elements of $\mathcal{SN}_\kappa$ from the ground model. Subsequently, we show that S_κ adds a generic κ-real that is not an element of any $X \in (\mathcal{SN}_\kappa)^{\mathbf{V}}$. This concludes the proof, since any κ^+-sized witness of $\mathrm{cov}(\mathcal{SN}_\kappa)$ obtained in an intermediate step of the iteration is destroyed by the subsequent step, and it is known that $\leq\kappa$-support iteration of S_κ does not increase $\mathrm{cof}(\mathcal{M}_\kappa)$ (see e.g. [BBTFM18]).

Since $\mathbf{V}$ “$2^\kappa = \kappa^+$”, we may fix some dominating family $D \subseteq {}^\kappa\kappa$ of size κ^+. Let $\sigma = \{\bar s^f \mid f \in D\}$ be a family of sequences, where $\bar s^f = \langle s^f_\xi \mid \xi \in \kappa\rangle$ is such that $s^f_\xi \in {}^{f(\xi)}2$. We use σ to define an element of $\mathcal{SN}_\kappa$:

$$\llbracket\sigma\rrbracket = \bigcap_{f\in D} \bigcap_{\alpha_0\in\kappa} \bigcup_{\xi\in[\alpha_0,\kappa)} [s^f_\xi] \in \mathcal{SN}_\kappa.$$

[16] A more complicated forcing notion is necessary, since the goal of the mentioned paper is not only to increase $\mathrm{cov}(\mathcal{SN}_\kappa)$, but also to increase $\mathrm{add}(\mathcal{SN}_\kappa)$. Sacks forcing could be seen as a special case of the forcing used in the paper, and is still sufficient for our purpose. The relevant idea is [GJS93, Lemma 2.25]. Generalising the more complicated proof of [GJS93] to the higher context has been done as well, by Schürz [CS].

Reversely it's easy to see that for every $X \in \mathcal{SN}_\kappa$ there exists some choice of σ such that $X \subseteq \mathsf{J}\sigma\mathsf{K}$. We show that the set of conditions $S \in \mathsf{S}_\kappa$ with $[S] \cap \mathsf{J}\sigma\mathsf{K} = ?$ is dense. It suffices to show that for each $S \in \mathsf{S}_\kappa$ there exists $f \in D$ and $T \leq S$ such that $[T] \cap \bigcup_{\xi\in\kappa}[s^f_\xi] = ?$.

Let $S \in \mathsf{S}_\kappa$ be arbitrary. We may assume without loss of generality (by pruning) that for each α there is β_α such that $\mathrm{Split}_\alpha(S) \subseteq {}^{\beta_\alpha}2$, and thus that $\langle \beta_\alpha \mid \alpha \in \kappa\rangle$ enumerates the club set of the heights of splitting levels of S. Now let γ_ξ denote the ξ-th limit ordinal below κ and consider the nonstationary set $\left\{\beta_{\gamma_\xi+1} \mid \xi \in \kappa\right\}$. Let $f_0 : \xi \mapsto \beta_{\gamma_\xi+1} + 1$ and pick $f \in D$ such that $f_0 \leq^* f$.

For each $\xi \in \kappa$ such that $f_0(\xi) \leq f(\xi)$ and $s^f_\xi \quad \beta_{\gamma_\xi+1} \in S$, we prune S by removing the part of the tree generated by the successor $s^f_\xi \quad (\beta_{\gamma_\xi+1}+1)$ to get a tree $T \leq S$. Since $f_0 \leq^* f$, it follows that $[T] \cap \mathsf{J}\sigma\mathsf{K} = ?$. Finally, we only prune S at successor splitting levels, thus $T \in \mathsf{S}_\kappa$. □

κ-Miller Forcing

Generalisations of Miller forcing to ${}^\kappa\kappa$ have been studied in [FZ10, FHZ13, BBTFM18]. All of the results in this subsection could be found in the above references. We will introduce κ-Miller forcing as a forcing to compare the forcing notions of Chapter 5 to, and to show that unbounded κ-reals and b-unbounded κ-reals are distinct kinds of κ-reals.

We first mention that there exist choices for $\mathcal{U} \subseteq \mathcal{P}(\kappa)$ such that $\mathsf{Mi}^{\mathcal{U}}_\kappa$ adds a κ-Cohen generic, for instance if $\mathcal{U}$ is the club filter:

Theorem 4.4.10 — *[BBTFM18, Proposition 77]*
Let $\mathcal{U}$ be the club filter on κ, that is, the set of all subsets of κ containing a club set. Then $\mathsf{Mi}^{\mathcal{U}}_\kappa$ adds a κ-Cohen generic.

On the other hand, if $\mathcal{U}$ is a $<\kappa$-complete normal ultrafilter (and hence κ is measurable), then we can show that $\mathsf{Mi}^{\mathcal{U}}_\kappa$ does not add κ-Cohen generics.

Theorem 4.4.11 — *[BBTFM18, Proposition 81]*
Let $\mathcal{U}$ be a $<\kappa$-complete normal ultrafilter on κ, then $\mathsf{Mi}^{\mathcal{U}}_\kappa$ has the pow-Laver property. ◁

Since the pow-Laver property also implies the (b, pow)-Laver property for any $b \in {}^\kappa\kappa$ with $\mathsf{pow} < \mathrm{cf}(b)$, we see by Lemma 4.2.12 that $\mathsf{Mi}^{\mathcal{U}}_\kappa$ does not add a b-unbounded κ-real when $\mathcal{U}$ is a $<\kappa$-complete ultrafilter. Therefore, $\mathsf{Mi}^{\mathcal{U}}_\kappa$ cannot add a κ-Cohen generic either.

Theorem 4.4.12 — *Folklore, see e.g. [BBTFM18, after Definition 74]*
$\mathsf{Mi}^{\mathcal{U}}_\kappa$ adds an unbounded κ-real. ◁

Proof. If $G \subseteq \mathsf{Mi}^{\mathcal{U}}_\kappa$ be a generic filter, then $\bigcap G \in {}^\kappa\kappa$. In the ground model, let $\dot{f}$ be a name for $\bigcap G$, let $g \in {}^\kappa\kappa$ and let $T_0 \in \mathsf{Mi}^{\mathcal{U}}_\kappa$.

We construct a fusion sequence $\langle T_\alpha \mid \alpha \in \kappa\rangle$. We obtain $T_{\alpha+1}$ from T_α by removing all of the nodes $v \in \mathrm{suc}(u, T_\alpha)$ with $u \in \mathrm{Split}_{\alpha+1}(T_\alpha)$ for which $v(\mathrm{dom}(u)) < g(\mathrm{dom}(u))$. Since $\mathcal{U}$ is a nonprincipal filter and we remove only $|g(\mathrm{dom}(u))| < \kappa$ many sets from $\mathrm{suc}(u, T_\alpha)$, we see that

$\{x \mid u^\frown\langle x\rangle \in T_{\alpha+1}\} \in \mathcal{U}$ remains true. Also note that $\mathrm{Split}_\alpha(T_\alpha) = \mathrm{Split}_\alpha(T_{\alpha+1})$, ensuring that $T_{\alpha+1} \leq_\alpha T_\alpha$. For limit α, we set $T_\alpha = \bigcap_{\xi<\alpha} T_\xi$.

If T is the fusion limit of $\langle T_\alpha \mid \alpha \in \kappa\rangle$ and $\alpha_0 \in \kappa$, then for any $u \in \mathrm{Split}_{\alpha_0+1}(T)$ we see that $\mathrm{dom}(u) \geq \alpha_0$ and $(T)_u$ "$g(\mathrm{dom}(u)) \leq \dot f(\mathrm{dom}(u))$". It follows that T "$g \leq^\infty \dot f$". □

This shows that $\mathsf{Mi}^{\mathcal{U}}_\kappa$ is not ${}^\kappa\kappa$-bounding, and thus also does not have the pow-Sacks property.

Theorem 4.4.13

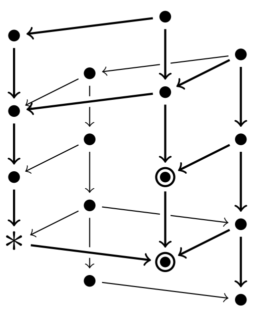
Let $\mathcal{U}$ be a $<\kappa$-complete filter on κ, then $\mathsf{Mi}^{\mathcal{U}}_\kappa$ does not add an eventually different κ-real. ◁

Proof. Let $T_0 \in \mathsf{Mi}^{\mathcal{U}}_\kappa$ be such that T_0 "$\dot f \in {}^\kappa\kappa$", then we will use fusion to construct a condition $T \leq T_0$ and a κ-real $g \in {}^\kappa\kappa$ such that T "$\dot f =^\infty g$". We let $\langle K_\alpha \mid \alpha \in \kappa\rangle$ be a partition of κ into sets of size κ, and we let $\langle k^\xi_\alpha \mid \xi \in \kappa\rangle$ enumerate K_α.

To define $T_{\alpha+1}$ from T_α, we let $\langle v_\xi \mid \xi \in \kappa\rangle$ be an enumeration of all $v \in T_\alpha$ such that $v = u^\frown\langle x\rangle$ for some $u \in \mathrm{Split}_\alpha(T_\alpha)$. This is possible by inaccessibility of κ and $T_\alpha \subseteq {}^{<\kappa}\kappa$, hence $|T_\alpha| = \kappa$.

For each $\xi \in \kappa$ we find some $T^\xi_\alpha \leq (T_\alpha)_{v_\xi}$ such that there exists $\beta_\xi \in \kappa$ with T^ξ_α "$\dot f(k^\xi_\alpha) = \check\beta_\xi$" and we define $g(k^\xi_\alpha) = \beta_\xi$.

The rest of the proof follows Theorem 4.4.8. □



A schematic representation of Figure 4.1, showing the effect of κ-Miller forcing guided by a $<\kappa$-complete normal ultrafilter with $\mathrm{cf}(\mathfrak{b}) > \mathsf{pow}$.

Legend
◉ added by the forcing
• not added by the forcing
∗ unknown for $h = \mathsf{id}$, not added for $h \geq^* \mathsf{pow}$

In Chapter 5 we will introduce a forcing notion that adds a $\mathfrak{b}$-unbounded κ-real without adding unbounded κ-reals.

κ-Laver Forcing

We saw in the last section that κ-Miller forcing could add unbounded κ-reals without adding κ-Cohen generics. A natural strengthening of this would be whether it is possible to add dominating κ-reals without adding κ-Cohen generics. Classically, this is what Laver forcing achieves, that is, Laver forcing adds a dominating real without adding Cohen reals.

The situation is very different on ${}^\kappa\kappa$, as was shown by Khomskii, Koelbing, Laguzzi & Wohofsky [KKLW22]. In this paper it is shown that κ-Laver forcing $\mathsf{L}^{\mathcal{U}}_\kappa$ will always add a κ-Cohen generic[17], as will many other $<\kappa$-distributive forcing notions similar to $\mathsf{L}^{\mathcal{U}}_\kappa$.

It also follows from this that κ-Laver forcing cannot have any Laver properties, because these prevent κ-Cohen generics from being added by Lemma 4.2.12.

[17] This is implied by [KKLW22, Theorem 3.5]

4.5. Open Questions

Reading through this chapter, it may be apparent that there are many holes left to be plugged, in the sense that we do not know if Figure 4.1 is complete, nor do we have a complete overview of which kinds of κ-reals are added by some of the mentioned forcing notion. We will not give a complete overview of all the open questions that could be deduced from this chapter, but we will mention some that are of particular (general) interest.

The problem we mention in the subsection about κ-Laver forcing is perhaps the most intriguing open problem, since it is in stark contrast with the classical analogy:

Question 4.5.1 — *[KKLW22, Question 5.1]*
Does there exist a $<\kappa$-distributive forcing notion that adds a dominating κ-real without adding a κ-Cohen generic? Does every $<\kappa$-closed forcing notion adding a dominating κ-real add a κ-Cohen generic? ◁

This question is related to Questions 2.6.1 and 2.6.2, since such a forcing might be usable to add many dominating κ-reals, thereby increasing $\mathsf{b}_\kappa(\leq^*)$, without adding κ-Cohen generics, which implies that $\mathrm{cov}(\mathcal{M}_\kappa) = \kappa^+$.

Specifically for Question 2.6.2 we may also attempt to find a forcing notion that could be iterated such that many unbounded κ-reals are added without adding κ-Cohen generics. As we saw, κ-Miller forcing has this property, but it is known that products of κ-Miller forcing add a κ-Cohen generic (see [BBTFM18, Theorem 85]), and for iterations it is unknown whether the Laver property is preserved.

Question 4.5.2 — *[BBTFM18, Question 83]*
Let $\overline{\mathbb{M}\mathrm{i}}$ be a $\leq\kappa$-support iteration of κ-Miller forcing guided by $<\kappa$-complete normal ultrafilters. Does $\overline{\mathbb{M}\mathrm{i}}$ have the pow-Laver property? ◁

Finally we will mention the question of whether it is consistent that $\mathsf{b}^h_\kappa(\in^*) \neq \mathsf{b}^{h'}_\kappa(\in^*)$ for some parameters h, h'. In the dual case we know that $\mathsf{d}^{\mathsf{pow}}_\kappa(\in^*) < \mathsf{d}^{\mathsf{id}}_\kappa(\in^*)$ is consistent, and as we will see in the next chapter, we can even separate κ^+ many localisation cardinals with different parameters. These consistency results are shown using forcing notions resembling κ-Sacks or κ-Miller forcing, which are not helpful in separating avoidance numbers $\mathsf{b}^h_\kappa(\in^*)$.

Perhaps it is possible to separate localisation and avoidance numbers by using h-κ-localisation forcing with specific parameters h, making the following question relevant:

Question 4.5.3
Does there exist h' such that $\mathbb{L}\mathrm{oc}^h_\kappa$ does not add an h'-localising κ-real? ◁

5

Lots and Lots of Localisation Cardinals

In this chapter we discuss how to separate many cardinalities of the form $\mathsf{d}^h_\kappa(\in^*)$. This chapter contains new results and could be seen as an extension of the previous results regarding $\mathsf{d}^h_\kappa(\in^*)$ found in [BBTFM18].

Remember from Fact 2.4.3 that $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ are characterised using $\mathsf{b}^h(\in^*)$ and $\mathsf{d}^h(\in^*)$ for any arbitrary cofinal $h \in {}^\omega\omega$. Therefore, the parameter h cannot influence the size of these cardinal characteristics in the classical context. On the other hand, it was proved in [BBTFM18] that $\mathsf{d}^h_\kappa(\in^*)$ can consistently have different values for different $h \in {}^\kappa\kappa$, namely $\mathsf{d}^{\mathsf{pow}}_\kappa(\in^*) < \mathsf{d}^{\mathsf{id}}_\kappa(\in^*)$ is consistent, since this inequality holds in the κ-Sacks model, essentially because S_κ has the pow-Sacks property, but not the id-Sacks property.

In this chapter we will answer an open question from [BBTFM18] and prove that there exist functions $h_\xi \in {}^\kappa\kappa$ and distinct cardinals λ_ξ with $\mathrm{cf}(\lambda_\xi) > \kappa$ for each $\xi \in \kappa^+$ such that it is simultaneously consistent that $\mathsf{d}^{h_\xi}_\kappa(\in^*) = \lambda_\xi$ for all $\xi \in \kappa$. The strategy will be the same as the strategy used to separate $\mathsf{d}^{\mathsf{id}}_\kappa(\in^*)$ from $\mathsf{d}^{\mathsf{pow}}_\kappa(\in^*)$, in that we consider a product of perfect tree forcing notions that have the h_ξ-Sacks property, but not the h_η-Sacks property for different ξ and η. Our candidates for these forcing notions are κ-Miller Lite forcing ML^h_κ, where the parameter h will play a key role.

In the first section, we introduce κ-Miller Lite forcing ML^h_κ. We will prove that it preserves cardinals and cofinalities and use fusion to show that it satisfies certain Sacks properties. In the second section we consider products of such forcing notions. We show that properties such as the preservation of cardinals and cofinalities and the relevant Sacks properties are preserved under $\leq\kappa$-support products and we use this to prove the consistency of κ many different cardinals. Finally, in the third section we will show that with a preparatory forcing, we can use our approach to prove the consistency of κ^+ many different cardinals.

Nota Bene! In this chapter we assume without mention that κ denotes an inaccessible cardinal. We will also fix the convention that $h, H, F \in {}^\kappa\kappa$ denote increasing cofinal cardinal functions. This also extends to indexed or accented functions using the symbols b, h, F, such as b_ξ, h', and so on.

5.1. κ-Miller Lite Forcing

In order to prove that κ-Sacks forcing S_κ has the pow-Sacks property, but not the id-Sacks property, one uses an argument by fusion and counts the minimum number of splitting nodes in $\mathrm{Split}_\alpha(T)$ for conditions $T \in \mathsf{S}_\kappa$. Essentially, the conditions split often enough to prevent a generic real from being id-localised, but the number of successors of splitting nodes (which is 2) is small enough to construct a pow-slalom that localises any name for a κ-real.

Another forcing we have considered is κ-Miller forcing, where each splitting node has κ many successors. This is too large to prove the h-Sacks property for any $h \in {}^{\kappa}\kappa$, since κ-Miller forcing adds an unbounded κ-real.

Our approach is therefore to describe a forcing notion that splits exactly often enough to fail the h-Sacks property, but not often enough to fail the h'-Sacks property, for two functions h, h'. We could consider a bounded version of κ-Miller forcing for this. On ${}^{\omega}\omega$, such a forcing has been studied in [Ges06] as ***Miller Lite forcing***, and we will adopt this name as well.

Definition 5.1.1
The conditions of the forcing ML^h_κ are closed perfect trees $T \subseteq {}^{<\kappa}\kappa$ that satisfy the *h-splitting property*: if $u \in \mathrm{Split}_\alpha(T)$, then u is an $h(\alpha)$-splitting node in T. The order is defined as $T \leq S$ iff $T \subseteq S$ and for any $u \in T$, if $\mathrm{suc}(u,T) \neq \mathrm{suc}(u,S)$, then $|\mathrm{suc}(u,T)| < |\mathrm{suc}(u,S)|$. $\triangleleft$

We naturally want ML^h_κ to preserve cardinalities. If we assume that $\mathbf{V}$ “$2^\kappa = \kappa^+$”, then it is clear that ML^h_κ has the $<\kappa^{++}$-chain condition, since $|\mathsf{ML}^h_\kappa| \leq |\mathcal{P}({}^{<\kappa}\kappa)| = 2^\kappa = \kappa^+$, where the former equality is implied by $\kappa^{<\kappa} = \kappa$, which in turn follows from κ being inaccessible. Therefore cardinalities above κ^+ are preserved under assumption of $\mathbf{V}$ “$2^\kappa = \kappa^+$”.

To preserve cardinalities less than or equal to κ, we show that ML^h_κ is $<\kappa$-closed.

Lemma 5.1.2
ML^h_κ is $<\kappa$-closed. In fact, if $\langle T_\xi \mid \xi \in \lambda\rangle$ is a descending sequence of conditions with $\lambda < \kappa$ limit, then $T = \bigcap_{\xi\in\lambda} T_\xi$ is a condition below each T_ξ. $\triangleleft$

Proof. We first prove:

($*$) if $u \in T = \bigcap_{\xi\in\lambda} T_\xi$, then there is $\eta \in \lambda$ such that $\mathrm{suc}(u,T) = \mathrm{suc}(u,T_\eta)$.

Suppose that $u \in T$, and let $\lambda_\xi = |\mathrm{suc}(u,T_\xi)|$, then the ordering on ML^h_κ dictates that $\langle \lambda_\xi \mid \xi \in \lambda\rangle$ is a descending sequence of cardinals, hence there is $\eta \in \lambda$ such that $\lambda_\xi = \lambda_\eta$ for all $\xi \in [\eta,\lambda)$. But then $\mathrm{suc}(u,T_\xi) = \mathrm{suc}(u,T_\eta)$ for all $\xi \in [\eta,\lambda)$ by the ordering of ML^h_κ.

We show that $T = \bigcap_{\xi\in\lambda} T_\xi$ satisfies the lemma by verifying that T is perfect, closed under splitting, satisfies the h-splitting property and that $T \leq T_\xi$ for all $\xi \in \lambda$.

Perfect. Let $u \in T$, and let $f \in [T]$ be a branch for which $u \subseteq f$. If $\mathrm{dom}(f) < \kappa$, then $f \in T_\xi$ for each ξ, thus by ($*$) there is some $\eta \in \lambda$ for which $\mathrm{suc}(f,T_\eta) = \mathrm{suc}(f,T) =$?. Then clearly $T_\eta \notin \mathsf{ML}^h_\kappa$, which is a contradiction, hence $\mathrm{dom}(f) = \kappa$.

Let $C_\xi = \{\alpha \in [\mathrm{dom}(u),\kappa) \mid f \quad \alpha$ is splitting in $T_\xi\}$, then since $T_\xi \in \mathsf{ML}^h_\kappa$ are conditions, we see that C_ξ is a club set. But then $\bigcap_{\xi\in\lambda} C_\xi$ is club. Any $v \in \bigcap_{\xi\in\lambda} C_\xi$ is splitting in all T_ξ, thus by ($*$) it is splitting in T. By definition of C_ξ it follows that $u \subseteq v$ holds for such $v \in \bigcap_{\xi\in\lambda} C_\xi$.

Closed under splitting. Let $C \subseteq T$ be a chain of splitting nodes, then for every $\xi \in \lambda$ we also see that C is a chain of splitting nodes in T_ξ, and thus $\bigcup C$ is a splitting node in all T_ξ, hence by $(*)$, $\bigcup C$ is splitting in T.

h-Splitting property. If $u \in \mathrm{Split}_\alpha(T)$, then by $(*)$ there is $\eta \in \lambda$ such that $\mathrm{suc}(u, T_\eta) = \mathrm{suc}(u, T)$. Therefore $u \in \mathrm{Split}_\beta(T_\eta)$ for some $\beta \geq \alpha$, hence u is an $h(\beta)$-splitting in T. Remember that we assume h is increasing, so u is also $h(\alpha)$-splitting in T.

Ordering. Clearly $T \subseteq T_\xi$ for each $\xi \in \lambda$, and if $u \in T$ and $\mathrm{suc}(u, T) \neq \mathrm{suc}(u, T_\xi)$, then by $(*)$ there exists $\eta \in \lambda$ such that $\mathrm{suc}(u, T) = \mathrm{suc}(u, T_\eta)$, and clearly $\xi < \eta$. Since $T_\eta \leq T_\xi$ by assumption, then $|\mathrm{suc}(u, T)| = |\mathrm{suc}(u, T_\eta)| < |\mathrm{suc}(u, T_\xi)|$. Hence $T \leq T_\xi$. □

Corollary 5.1.3

ML^h_κ preserves all cardinalities and cofinalities $\leq \kappa$.

What is left, is to show that κ^+ is also preserved. This will be a consequence of the proof that ML^h_κ has the F-Sacks property for some suitably large $F \in {}^\kappa\kappa$, so we will prove this first. But before that, we will need to show that ML^h_κ is closed under fusion. Our fusion ordering $\langle \leq_\alpha \mid \alpha \in \kappa \rangle$ will be the one from Definition 4.4.5, that is, $T \leq_\alpha S$ iff $T \leq S$ and $\mathrm{Split}_\alpha(T) = \mathrm{Split}_\alpha(S)$.

Lemma 5.1.4

If $\langle T_\alpha \mid \alpha \in \kappa \rangle$ is a fusion sequence, then $T = \bigcap_{\alpha \in \kappa} T_\alpha \in \mathsf{ML}^h_\kappa$ and $T \leq_\alpha T_\alpha$ for all $\alpha \in \kappa$. ◁

Proof. That T is a closed perfect tree follows as in Lemma 4.4.6. We will show that T has the h-splitting property and that $T \leq_\alpha T_\alpha$ for all $\alpha \in \kappa$.

h-Splitting property. Let $u \in \mathrm{Split}_\alpha(T)$, then u is an $h(\alpha)$-splitting node in $T_{\alpha+1}$. Let $\lambda_u = |\mathrm{suc}(u, T_{\alpha+1})| \geq h(\alpha)$ and let $\langle v_\xi \mid \xi \in \lambda_u \rangle$ enumerate those $v \supseteq u$ such that $v \in \mathrm{Split}_{\alpha+1}(T_{\alpha+1})$. For all $\beta \geq \alpha + 1$ we have $\mathrm{Split}_{\alpha+1}(T_\beta) = \mathrm{Split}_{\alpha+1}(T_{\alpha+1})$, therefore for each $\xi \in \lambda_u$ we see that $v_\xi \in T_\beta$ for all $\beta > \alpha$, thus $v_\xi \in T$. Therefore u is $h(\alpha)$-splitting in T.

Ordering. Clearly $T \subseteq T_\alpha$ for all $\alpha \in \kappa$. Given $u \in T$ and $\alpha \in \kappa$ such that $\mathrm{suc}(u, T) \neq \mathrm{suc}(u, T_\alpha)$, we will show that $|\mathrm{suc}(u, T)| < |\mathrm{suc}(u, T_\alpha)|$. We may assume without loss of generality that u is splitting in T, so let $\beta \in \kappa$ be such that $u \in \mathrm{Split}_\beta(T)$. Since $\mathrm{Split}_\gamma(T) = \mathrm{Split}_\gamma(T_\alpha)$ for all $\gamma \leq \alpha$, we see that $\beta \geq \alpha$. We have $\mathrm{Split}_{\beta+1}(T_{\beta+1}) = \mathrm{Split}_{\beta+1}(T)$, and thus $\mathrm{suc}(u, T_{\beta+1}) = \mathrm{suc}(u, T)$. Finally $T_{\beta+1} \leq T_\alpha$ gives us $|\mathrm{suc}(u, T)| = |\mathrm{suc}(u, T_{\beta+1})| < |\mathrm{suc}(u, T_\alpha)|$. □

We are now ready to prove the two main ingredients necessary for separating the localisation cardinals. We will show that for any h there is some faster growing F such that ML^h_κ has the F-Sacks property, and reversely that for any F there exists some faster growing h such that ML^h_κ does not have the F-Sacks property. In other words, for any F_0 we may find h and F_1 such that ML^h_κ does not have the F_0-Sacks property, but does have the F_1-Sacks property. It will be helpful to establish the notion of sharp trees.

Let a κ-tree $T \in \mathsf{ML}^h_\kappa$ be called ***sharp*** if every $u \in \mathrm{Split}_\alpha(T)$ is a sharp $h(\alpha)$-splitting node. It is clear that by pruning we may find a sharp T^* below any condition $T \in \mathsf{ML}^h_\kappa$ such that

$\operatorname{Split}_\alpha(T^*) \subseteq \operatorname{Split}_\alpha(T)$ for every $\alpha \in \kappa$. We may assume that we can canonically do so, thus we will hereby fix the notation T^* to denote a canonical sharp κ-tree below condition T. We will write $(\mathsf{ML}^h_\kappa)^* = \{T \in \mathsf{ML}^h_\kappa \mid T \text{ is sharp}\}$, which embeds densely into ML^h_κ.

Remember that $(T)_u$ is the subtree generated by $u \in T$.

Theorem 5.1.5

For any h there exists F such that $h <^* F$ and ML^h_κ has the F-Sacks property. ◁

Proof. We will let $F : \alpha \mapsto (h(\alpha)^{|\alpha|})^+$ and show that ML^h_κ has the F-Sacks property.

Let $T_0 \in \mathsf{ML}^h_\kappa$ and let $\dot f$ be a name such that $T_0 \quad$ "$\dot f \in {}^\kappa\kappa$". If $T_0 \quad$ "$\dot f \in \mathbf{V}$", then the existence of an appropriate F-slalom is obvious, so we assume that $T_0 \quad$ "$\dot f \notin \mathbf{V}$". We will construct a fusion sequence $\langle T_\xi \mid \xi \in \kappa\rangle$ and sets $\{D_\xi \subseteq \kappa \mid \xi \in \kappa\}$ with $|D_\xi| < F(\xi)$ such that $\bigcap_{\xi\in\kappa} T_\xi = T \quad$ "$\dot f(\xi) \in \check D_\xi$" for each $\xi \in \kappa$. Consequently, we can define the F-slalom $\varphi : \xi \mapsto D_\xi$ in the ground model, then it follows that $T \quad$ "$\dot f(\xi) \in \check\varphi(\xi)$" for all $\xi \in \kappa$.

In general, we will assume each T_ξ has the following property:

$(*)$ for every $u \in \operatorname{Split}_\alpha(T_\xi)$ with $\alpha < \xi$ we have $|\operatorname{suc}(u, T_\xi)| = h(\alpha)$.

This is vacuously true for T_0, and by using sharp κ-trees at successor stages of our construction, $(*)$ will follow by induction. If γ is limit, we will let $T_\gamma = \bigcap_{\xi\in\gamma} T_\xi$, which is a condition by Lemma 5.1.2. T_γ need not necessarily be a sharp κ-tree, but it is at least sharp for all splitting levels less than γ, which is enough for $(*)$.

Suppose T_ξ has been defined, then we will define $T_{\xi+1}$ such that it limits the possible values of $\dot f(\xi)$ and such that $T_{\xi+1} \le_\xi T_\xi$. First note that if T_ξ has property $(*)$, then $T^*_\xi \le_\xi T_\xi$: If u is splitting in T_ξ and $u \notin T^*_\xi$, then u was removed because there is some $v \subseteq u$ such that $\operatorname{suc}(v, T_\xi)$ is too large for sharpness. But then by $(*)$ it follows that $v \in \operatorname{Split}_\alpha(T_\xi)$ for some $\alpha \ge \xi$, hence $u \in \operatorname{Split}_\beta(T_\xi)$ for some $\beta > \xi$.

We define a set V_ξ of successor nodes of the ξ-th splitting level, that is:

$$V_\xi = \bigcup \{\operatorname{suc}(u, T^*_\xi) \mid u \in \operatorname{Split}_\xi(T^*_\xi)\}.$$

Our goal is to find a stronger condition below each subtree $(T^*_\xi)_v$ with $v \in V_\xi$ that decides $\dot f(\xi)$, and glue these conditions back together to get a condition stronger than T^*_ξ. Since the size of V_ξ is limited, this limits the possible values of $\dot f(\xi)$ to a small set.

For each $v \in V_\xi$ find a condition $T^v \le (T^*_\xi)_v$ such that $T^v \quad$ "$\dot f(\xi) = \check\beta^v_\xi$" for some $\beta^v_\xi \in \kappa$. Choose some arbitrary $u \in \operatorname{Split}_\xi(T^v)$ and $w \in \operatorname{suc}(u, T^v)$, and consider the subtree $(T^v)_w$ of T^v generated by the initial segment w. We let $G_\xi : V_\xi \to \mathcal{P}(T_\xi)$ send $v \mapsto (T^v)_w$. Note that the α-th splitting level of $G_\xi(v) = (T^v)_w$ corresponds to the $(\xi + 1 + \alpha)$-th splitting level of T^v.

Now we define:

$$T_{\xi+1} = \bigcup G_\xi[V_\xi] = \bigcup \{G_\xi(v) \mid v \in V_\xi\},$$
$$D_\xi = \{\beta^v_\xi \mid v \in V_\xi\}.$$

For each $v \in V_\xi$ we have $v \in G_\xi(v)$, thus each successor of a splitting node in $\mathrm{Split}_\xi(T_\xi)$ is in $T_{\xi+1}$. Therefore we see that $\mathrm{Split}_\xi(T_{\xi+1}) = \mathrm{Split}_\xi(T_\xi)$. If $u \in \mathrm{Split}_{\xi+1+\alpha}(T_{\xi+1})$ for some $\alpha \in \kappa$, then $u \in \mathrm{Split}_\alpha(G_\xi(v))$, thus $u \in \mathrm{Split}_{\xi+1+\alpha}(T^v)$, and since $T^v \in \mathsf{ML}^h_\kappa$, we see that u is $h(\xi+1+\alpha)$-splitting. Therefore $T_{\xi+1}$ satisfies the h-splitting property of Definition 5.1.1. It is easy to check that $T_{\xi+1}$ is a closed perfect tree, thus we can conclude that $T_{\xi+1} \in \mathsf{ML}^h_\kappa$ and that $T_{\xi+1} \leq_\xi T_\xi$.

Note that the set D_ξ is indeed small enough:

$$|D_\xi| \leq |V_\xi| = |\mathrm{Split}_\xi(T^*_\xi)| \cdot h(\xi) \leq h(\xi)^{|\xi|} < F(\xi).$$

For each $v \in V_\xi$ we have T^v “$\dot{f}(\xi) \in \check{D}_\xi$”, and $\{T^v \mid v \in V_\xi\}$ is a maximal antichain below $T_{\xi+1}$, thus:

$$T_{\xi+1} \quad \text{“}\dot{f}(\xi) \in \check{D}_\xi\text{”}.$$

Let $T = \bigcap_{\xi\in\kappa} T_\xi$, then by Lemma 5.1.4 we see $T \in \mathsf{ML}^h_\kappa$, and T “$\dot{f}(\xi) \in \check{D}_\xi$” for all $\xi \in \kappa$. □

As a corollary of ML^h_κ having the F-Sacks property, we immediately get that κ^+ is preserved.

Corollary 5.1.6
ML^h_κ preserves κ^+. ◁

Proof. Given an ML^h_κ-name $\dot{f}$ and $T \in \mathsf{ML}^h_\kappa$ such that T “$\dot{f} : \kappa \to \kappa^+$”, then using (the proof of) the F-Sacks property we may produce sets D_ξ with $|D_\xi| < F(\xi) < \kappa$ for each $\xi \in \kappa$ such that T' “$\dot{f}(\xi) \in \check{D}_\xi$” for some stronger $T' \leq T$, and thus $\dot{f}$ is forced to have a range contained in $\bigcup_{\xi\in\kappa} D_\xi$ and cannot be cofinal in κ^+. □

The second ingredient is to find a suitably fast growing h for a given function F such that ML^h_κ does not have the F-Sacks property. We will need the following lemma.

Lemma 5.1.7
Let $T \in \mathsf{ML}^h_\kappa$ and let $C_T = \{\alpha \in \kappa \mid \mathrm{Split}_\alpha(T) = T \cap {}^\alpha\kappa\}$, then C_T is a club set. ◁

Proof. For $\alpha_0 \in \kappa$ we can recursively define α_{n+1} large enough such that $\mathrm{Split}_{\alpha_n}(T) \subseteq {}^{\leq\alpha_{n+1}}\kappa$ for each $n \in \omega$. Let $\alpha = \bigcup_{n\in\omega} \alpha_n$, then $\alpha \in C_T$, hence C_T is unbounded. It is easy to see that C_T is continuous. □

Theorem 5.1.8
Let $F \in {}^\kappa\kappa$, then there exists h such that ML^h_κ adds an F-avoiding κ-real, and hence does not have the F-Sacks property. ◁

Proof. Let h be such that $F(\alpha) \leq h(\alpha)$ for all $\alpha \in S$, where S is a stationary subset of κ. We will show that the ML^h_κ-generic κ-real is F-avoiding.

Let $\dot{f}$ be a name for the generic ML^h_κ-real in ${}^\kappa\kappa$, let φ be an F-slalom, let $T \in \mathsf{ML}^h_\kappa$ and let $\alpha_0 \in \kappa$. We want to find some $\alpha \geq \alpha_0$ and $S \leq T$ such that S “$\dot{f}(\alpha) \notin \check{\varphi}(\alpha)$”. If we can find $u \in T \cap {}^{\alpha+1}\kappa$ such that $u(\alpha) \notin \varphi(\alpha)$, then $(T)_u$ will be sufficient.

Let C_T be as defined in Lemma 5.1.7 and $\alpha \in C_T \cap S$ such that $\alpha_0 \le \alpha$, then $\mathrm{Split}_\alpha(T) = T \cap {}^\alpha\kappa$, thus each $t \in T \cap {}^\alpha\kappa$ is an $h(\alpha)$-splitting node. Hence, there is a set $X \subseteq \kappa$ with $|X| = h(\alpha)$ such that $t^\frown\langle\gamma\rangle \in T$ for all $\gamma \in X$. Since $|\varphi(\alpha)| < F(\alpha) \le h(\alpha)$, there is some $\gamma \in X$ such that $\gamma \notin \varphi(\alpha)$, and thus $u = t^\frown\langle\gamma\rangle$ is as desired. □

Theorem 5.1.9

ML^h_κ does not add eventually different κ-reals. ◁

Proof. The proof is as in Theorems 4.4.8 and 4.4.13. □

It follows that the situation is very comparable to that of κ-Sacks forcing, with h taking the role of id and $F : \alpha \mapsto (h(\alpha)^{|\alpha|})^+$ taking the role of pow.

For later use, we need to consider the relation of forcing notions with parameters that are almost equal. For functions $f, g \in {}^\kappa\kappa$, we say that f and g are ***almost equal*** if $f =^* g$, that is, if there exists $\xi \in \kappa$ such that $f(\alpha) = g(\alpha)$ for all $\alpha \in [\xi, \kappa)$.

Lemma 5.1.10

If $h =^* h'$, then ML^h_κ and $\mathsf{ML}^{h'}_\kappa$ are forcing equivalent. ◁

Proof. Since $\mathsf{ML}^h_\kappa \cap \mathsf{ML}^{h'}_\kappa$ is dense in both ML^h_κ and $\mathsf{ML}^{h'}_\kappa$. □

Finally we will briefly consider a variant on the forcing notion ML^h_κ defined in Definition 5.1.1, namely the bounded forcing $\mathsf{ML}^{b,h}_\kappa$ consisting of closed perfect trees on $\prod_{<\kappa} b$ that are h-splitting. It is easy to see that $\mathsf{ML}^{b,h}_\kappa$ will affect $\mathsf{d}^{b,F}_\kappa(\in^*)$ in the same way that ML^h_κ affects $\mathsf{d}^F_\kappa(\in^*)$ for any F with $F < \mathrm{cf}(b)$. The other thing that one should note is that ML^h_κ and $\mathsf{ML}^{b,h}_\kappa$ are in fact forcing equivalent if $h(\alpha)^{|\alpha|} \le b(\alpha)$ for all $\alpha \in \kappa$: essentially one could relabel the nodes in $T \in (\mathsf{ML}^h_\kappa)^*$ to obtain a condition in $\mathsf{ML}^{b,h}_\kappa$, from which it is not hard to construct a dense embedding. Consequently, the results proved in this chapter also imply similar results for the bounded localisation numbers $\mathsf{d}^{b,h}_\kappa(\in^*)$.

5.2. Products of κ-Miller Lite Forcing

We see that for any F_0, we can find a faster growing F_1 and some suitable h such that the forcing ML^h_κ has the F_1-Sacks property and not the F_0-Sacks property, thus forcing with ML^h_κ will not increase $\mathsf{d}^{F_1}_\kappa(\in^*)$, but has the potential to increase $\mathsf{d}^{F_0}_\kappa(\in^*)$.

In order to increase $\mathsf{d}^{F_0}_\kappa(\in^*)$ we will need to add many ML^h_κ-generic κ-reals to the ground model, since these are F_0-avoiding κ-reals. This can be either done with an iteration, or with a product. Iteration has the drawback that once we have forced 2^κ to be of size κ^{++}, the forcing ML^h_κ no longer has the $<\kappa^{++}$-c.c., and thus we cannot sufficiently control the iteration past this point. While this does not form a problem to prove the consistency of $\kappa^+ = \mathsf{d}^{F_1}_\kappa(\in^*) < \mathsf{d}^{F_0}_\kappa(\in^*) = \kappa^{++}$, iteration proves to be an obstacle when we wish to force localisation cardinals to be larger than κ^{++}. In particular, our goal to simultaneously assign multiple localisation cardinals to different cardinalities requires a product.

Let us fix a set of ordinals $\mathcal{A}$ and parameters $\langle h_\xi \mid \xi \in \mathcal{A}\rangle$ for the forcing notions $\mathsf{ML}_\kappa^{h_\xi}$. For the remainder of this section we fix the $\leq\kappa$-support product $\overline{\mathsf{ML}} = \prod_{\xi\in\mathcal{A}}^{\leq\kappa} \mathsf{ML}_\kappa^{h_\xi}$. If $p, q \in \overline{\mathsf{ML}}$, we will often write $q \leq p$ instead of $q \leq_{\overline{\mathsf{ML}}} p$.

Lemma 5.2.1
$\overline{\mathsf{ML}}$ is $<\kappa$-closed.

Proof. By Theorem 4.1.15. □

We will also need a generalisation of the fusion lemma to work on product forcing. The generalisation of fusion described here is analogous to what is described in [Kan80] or [BBTFM18] and that we defined in Section 1.2. We define fusion orderings $\langle \leq_\alpha \mid \alpha \in \kappa\rangle$ on each $\mathsf{ML}_\kappa^{h_\xi}$ as in Lemma 5.1.4, which yields a generalised fusion ordering $\langle \leq_{Z,\alpha} \mid \alpha \in \kappa, Z \in [\mathcal{A}]^{<\kappa}\rangle$ on $\overline{\mathsf{ML}}$.

Lemma 5.2.2
If $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa\rangle$ is a generalised fusion sequence, then $\bigwedge_{\alpha\in\kappa} p_\alpha \in \overline{\mathsf{ML}}$. ◁

Proof. Suppose that $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa\rangle$ is a generalised fusion sequence, and let $p = \bigwedge_{\alpha\in\kappa} p_\alpha$. By definition of generalised fusion, every $\xi \in \operatorname{supp}(p) = \bigcup_{\alpha\in\kappa} p_\alpha$ is an element of Z_{η_ξ} for some $\eta_\xi \in \kappa$. This means that if $\beta \geq \alpha \geq \eta_\xi$, then $p_\beta(\xi) \leq_\alpha p_\alpha(\xi)$, and thus $\langle p_\alpha(\xi) \mid \alpha > \eta_\xi\rangle$ is a fusion sequence in $\mathsf{ML}_\kappa^{h_\xi}$. Since $\mathsf{ML}_\kappa^{h_\xi}$ is closed under fusion sequences (Lemma 5.1.4), we can conclude that

$$p(\xi) = \bigcap_{\alpha\in\kappa} p_\alpha(\xi) \in \mathsf{ML}_\kappa^{h_\xi}.$$

Since $\operatorname{supp}(p) = \bigcup_{\alpha\in\kappa} Z_\alpha$, we see that $|\operatorname{supp}(p)| \leq \kappa$, thus we can conclude that $p \in \overline{\mathsf{ML}}$. □

By Lemma 5.2.1, $\overline{\mathsf{ML}}$ preserves all cardinalities up to and including κ. Suppose that each $\mathsf{ML}_\kappa^{h_\xi}$ has the F-Sacks property for some suitably large F. We will show in the next lemma that this implies that $\overline{\mathsf{ML}}$ has the F-Sacks property and therefore preserves κ^+. Finally, if we assume that $\mathbf{V}$ “$2^\kappa = \kappa^+$”, then Theorem 4.1.17 shows that $\overline{\mathsf{ML}}$ is $<\kappa^{++}$-c.c. as well. Thus, $\overline{\mathsf{ML}}$ preserves all cardinals and cofinalities assuming that there exists some fixed $F \in {}^\kappa\kappa$ such that each $\mathsf{ML}_\kappa^{h_\xi}$ has the F-Sacks property.

Lemma 5.2.3
Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of ordinals and $\mathcal{B}^c = \mathcal{A} \setminus \mathcal{B}$, and consider a sequence of functions $\langle h_\xi \mid \xi \in \mathcal{A}\rangle$. We define the $\leq\kappa$-support product $\overline{\mathsf{ML}} = \prod_{\xi\in\mathcal{A}}^{\leq\kappa} \mathsf{ML}_\kappa^{h_\xi}$ and assume G is an $\overline{\mathsf{ML}}$-generic filter. If there exists $F \in {}^\kappa\kappa$ such that $(\sup_{\xi\in\mathcal{B}^c} h_\xi(\alpha))^{|\alpha|} < F(\alpha)$ for almost all $\alpha \in \kappa$, then for each $f \in ({}^\kappa\kappa)^{\mathbf{V}[G]}$ there is $\varphi \in (\mathrm{Loc}_\kappa^F)^{\mathbf{V}[G \upharpoonright \mathcal{B}]}$ such that $f \in^* \varphi$. ◁

Proof. Note that Lemma 5.1.10 implies that we can assume without loss of generality that $F(\alpha) > (\sup_{\xi\in\mathcal{B}^c} h_\xi(\alpha))^{|\alpha|}$ for all $\alpha \in \kappa$. Let $p \in \overline{\mathsf{ML}}$ and $\dot f$ be a name such that $p \;_{\overline{\mathsf{ML}}}$ “$\dot f \in {}^\kappa\kappa$”, then we will construct a name $\dot\varphi$ and a condition $p' \leq p$ such that p' “$\dot\varphi \in (\mathrm{Loc}_\kappa^F)^{\mathbf{V}[\dot G \upharpoonright \mathcal{B}]}$”.

The proof is essentially the same as the proof of Theorem 5.1.5, except that we work with generalised fusion sequences and have to construct a name $\dot\varphi$ for the appropriate F-slalom in

$\mathbf{V}[G \quad \mathcal{B}]$, since such a slalom is not generally present in the ground model. That is, we will construct a sequence $\langle (p_\xi, Z_\xi) \mid \xi \in \kappa \rangle$ with each $p_\xi \in \overline{\mathsf{ML}}$ that is a generalised fusion sequence in $\overline{\mathsf{ML}}$ and names $\dot{D}_\xi$ for sets of ordinals $D_\xi \in \mathbf{V}[G \quad \mathcal{B}]$ with $|D_\xi| < F(\xi)$, such that $p_{\xi+1} \quad$ "$\dot{f}(\xi) \in \dot{D}_\xi$".

For each $\xi \in \kappa$ and $\beta \in Z_\xi$ we will make sure that $p_\xi(\beta) \in (\mathsf{ML}_\kappa^{h_\beta})^*$ is sharp. To start, we let $p_0 = p$ and we let $Z_0 = \textbf{?}$. At limit stages δ we can define $p'_\delta = \bigwedge_{\xi \in \delta} p_\xi$ and let $p_\delta \leq p'_\delta$ be defined elementwise such that $p_\delta(\beta) = (p'_\delta(\beta))^*$ is sharp for each $\beta \in Z_\delta$.

Suppose we have defined $p_\xi \in \overline{\mathsf{ML}}$ and Z_ξ and that $|Z_\xi| \leq |\xi|$. As in the proof of Theorem 5.1.5, we will consider the successor nodes of the ξ-th splitting level, find subtrees that decide the value of $\dot{f}(\xi)$, and glue the subtrees together. However, in this situation we have to deal with multiple trees at once, namely with each $p_\xi(\beta)$ such that $\beta \in Z_\xi$. For each $\beta \in Z_\xi$ we define the set of successor nodes of the ξ-th splitting level of $p_\xi(\beta)$:

$$V_\xi^\beta = \bigcup \left\{ \operatorname{suc}(u, p_\xi(\beta)) \;\middle|\; u \in \operatorname{Split}_\xi(p_\xi(\beta)) \right\}.$$

To deal with $p_\xi(\beta)$ for all $\beta \in Z_\xi$ simultaneously, we have to consider combinations of elements of V_ξ^β for $\beta \in Z_\xi$, and for each combination we will define a condition that decides $\dot{f}(\xi)$. These combinations are given by functions $g : Z_\xi \to \bigcup_{\beta \in Z_\xi} V_\xi^\beta$ with the property that $g(\beta) \in V_\xi^\beta$. We will refer to such g as ***choice functions***, since g chooses an element of V_ξ^β for each $\beta \in Z_\xi$.

Let $\mathcal{V}_\xi$ be the set of choice functions on $\{V_\xi^\beta \mid \beta \in Z_\xi\}$ and $\mathcal{V}'_\xi$ the set of choice functions on $\{V_\xi^\beta \mid \beta \in Z_\xi \setminus \mathcal{B}\}$, that is, $\mathcal{V}'_\xi$ is the set of $g \quad (Z_\xi \setminus \mathcal{B})$ with $g \in \mathcal{V}_\xi$.

By induction hypothesis $p_\xi(\beta) \in (\mathsf{ML}_\kappa^{h_\beta})^*$ for each $\beta \in Z_\xi \setminus \mathcal{B}$, hence $|\operatorname{Split}_\xi(p_\xi(\beta))| = h_\beta(\xi)^{|\xi|}$ for all $\beta \in Z_\xi \setminus \mathcal{B}$ and thus, using that $|Z_\xi| \leq |\xi|$, we get

$$|\mathcal{V}'_\xi| \leq (\sup_{\beta \in Z_\xi \setminus \mathcal{B}} h_\beta(\xi)^{|\xi|})^{|Z_\xi \setminus \mathcal{B}|} \leq (\sup_{\beta \in \mathcal{B}^c} h_\beta(\xi))^{|\xi|} < F(\xi).$$

Therefore, if we restrict our attention to $Z_\xi \setminus \mathcal{B}$, the number of choice functions is small enough. Consequently, we can describe a name $\dot{D}_\xi$ depending only on the support in $\mathcal{B}$, i.e. $\dot{D}_\xi$ names a set in $\mathbf{V}[G \quad \mathcal{B}]$, such that $\dot{D}_\xi$ is bounded in cardinality by $F(\xi)$.

For any choice function $g \in \mathcal{V}_\xi$, let $(p_\xi)_g$ be the condition defined by

$$(p_\xi)_g(\beta) = \begin{cases} p_\xi(\beta) & \text{if } \beta \notin Z_\xi, \\ (p_\xi(\beta))_{g(\beta)} & \text{if } \beta \in Z_\xi. \end{cases}$$

Here $(p_\xi(\beta))_{g(\beta)}$ is the subtree of $p_\xi(\beta)$ generated by the initial segment $g(\beta) \in V_\xi^\beta$.

Let $\zeta = |\mathcal{V}_\xi|$ then $\zeta < \kappa$ by inaccessibility of κ. Fix some enumeration $\langle g_\eta \mid \eta \in \zeta \rangle$ of $\mathcal{V}_\xi$, which we will use to recursively define a decreasing sequence of conditions r_η with $r_\eta \leq_{Z_\xi, \xi} p_\xi$ for each $\eta \in \zeta$. Essentially, our recursive construction will result in $r_{\eta+1}$ being like r_η, except that $(r_\eta)_{g_\eta}$ is replaced by a stronger condition that decides $\dot{f}(\xi)$. At the end of the recursion, we will be left with a condition r_ζ such that $(r_\zeta)_g$ decides $\dot{f}(\xi)$ for every $g \in \mathcal{V}_\xi$. We then gather the possible values of $\dot{f}(\xi)$ to construct the name $\dot{D}_\xi$.

Let $r_0 = p_\xi$. For limit $\delta \in \zeta$ let $r_\delta = \bigwedge_{\eta\in\delta} r_\eta$, which is a condition by $<\kappa$-closure (Lemma 5.2.1). Assuming that $r_\eta \leq_{Z_\xi,\xi} p_\xi$ for each $\eta \in \delta$, it is easy to see that $r_\delta \leq_{Z_\xi,\xi} p_\xi$ as well.

Suppose r_η is defined and $r_\eta \leq_{Z_\xi,\xi} p_\xi$, then in particular $r_\eta(\beta) \leq_\xi p_\xi(\beta)$ for all $\beta \in Z_\xi$, and thus $\mathrm{Split}_\xi(r_\eta(\beta)) = \mathrm{Split}_\xi(p_\xi(\beta))$ for all $\beta \in Z_\xi$. Therefore by definition of the ordering on $\mathsf{ML}_\kappa^{h_\beta}$ and the fact that $p_\xi(\beta)$ is sharp, we see that V_ξ^β is exactly the set of successors of nodes at the ξ-th splitting level of $r_\eta(\beta)$. Take the η-th choice function $g_\eta \in \mathcal{V}_\xi$, and let $r'_\eta \leq (r_\eta)_{g_\eta}$ be such that r'_η “ $\dot{f}(\xi) = \check{\beta}_\xi^\eta$ ” for some ordinal β_ξ^η. We define $r_{\eta+1}$ elementwise.

If $\beta \notin Z_\xi$, then we simply take $r_{\eta+1}(\beta) = r'_\eta(\beta)$.

If $\beta \in Z_\xi$, fix some $w \in \mathrm{suc}(u, r'_\eta(\beta))$ for some $u \in \mathrm{Split}_\xi(r'_\eta(\beta))$ and consider the subtree $(r'_\eta(\beta))_w$ generated by the initial segment w. Now we are ready to define $r_{\eta+1}(\beta)$ as

$$r_{\eta+1}(\beta) = (r'_\eta(\beta))_w \;\cup\; \left\{ u \in r_\eta(\beta) \;\middle|\; \exists v \in V_\xi^\beta \setminus \{g_\eta(\beta)\}\ (u \subseteq v \text{ or } v \subseteq u) \right\}.$$

In words, $r_{\eta+1}(\beta)$ is the result of replacing the extensions of $g_\eta(\beta) \in r_\eta(\beta)$ by $(r'_\eta(\beta))_w$ that decides $\dot{f}(\xi)$, where we use the subtree $(r'_\eta(\beta))_w$ instead of $r'_\eta(\beta)$ to make sure that $r_{\eta+1}(\beta)$ has enough successors at each splitting level to be in $\mathsf{ML}_\kappa^{h_\beta}$ (compare this to the role of $(T^v)_w$ instead of T^v in the proof of Theorem 5.1.5).

To finish the construction of the next condition in the fusion sequence, we use $<\kappa$-closure to define $p'_{\xi+1} = \bigwedge_{\eta\in\zeta} r_\eta$ and let $p_{\xi+1} = (p'_{\xi+1})^*$ be sharp. To see that $p_{\xi+1} \leq_{Z_\xi,\xi} p_\xi$, note that for every $\beta \in Z_\xi$ and $v \in V_\xi^\beta$ we have $v \in r_\eta(\beta)$ for all $\eta \in \zeta$, hence $v \in p_{\xi+1}(\beta)$. This implies by definition of V_ξ^β that $p_{\xi+1}(\beta) \leq_\xi p_\xi(\beta)$ for all $\beta \in Z_\xi$. Finally, we can let $Z_{\xi+1} = Z_\xi \cup \{\delta\}$ for some ordinal δ, using bookkeeping to make sure that $\bigcup_{\xi\in\kappa} Z_\xi = \bigcup_{\xi\in\kappa} \mathrm{supp}(p_\xi)$.

Note that the set of conditions $r \leq p_{\xi+1}$ with $|r(\beta) \cap V_\xi^\beta| = 1$ for all $\beta \in Z_\xi$, is dense below $p_{\xi+1}$. For any such r, let g map β to the unique element of $r(\beta) \cap V_\xi^\beta$ for each $\beta \in Z_\xi$, then $g \in \mathcal{V}_\xi$ is a choice function, so we see that there exists $\eta \in \zeta$ such that $g = g_\eta$. We will show that $r \leq r'_\eta$, which implies that r “ $\dot{f}(\xi) = \check{\beta}_\xi^\eta$ ”.

For any β we have $r(\beta) \leq p_{\xi+1}(\beta) \leq r_{\eta+1}(\beta)$. If $\beta \notin Z_\xi$, then we simply have $r_{\eta+1}(\beta) = r'_\eta(\beta)$, thus we are done. Otherwise $\beta \in Z_\xi$, and we know that $g(\beta)$ is an initial segment of the stem of $r(\beta)$, hence

$$r(\beta) = (r(\beta))_{g(\beta)} \subseteq (r_{\eta+1}(\beta))_{g(\beta)} = (r'_\eta(\beta))_w,$$

where w is as in the definition of $r_{\eta+1}(\beta)$ above. Since $r(\beta) \leq r_{\eta+1}(\beta)$, we also have

$$r(\beta) = (r(\beta))_w \leq (r_{\eta+1}(\beta))_w = (r'_\eta(\beta))_w \leq r'_\eta(\beta),$$

and thus $r(\beta) \leq r'_\eta(\beta)$.

We are now ready to construct the names $\dot{D}_\xi$ such that:

$$p_{\xi+1} \quad \text{“}\, \dot{f}(\xi) \in \dot{D}_\xi \text{ and } \dot{D}_\xi \in \mathbf{V}[\dot{G} \quad \mathcal{B}] \text{ and } |\dot{D}_\xi| < F(\xi)\,\text{”}.$$

For any $g \in \mathcal{V}_\xi$, we define:

$$\begin{aligned} g'' &= g \quad (Z_\xi \cap \mathcal{B}), \\ E_g &= \{\eta \in \zeta \mid \exists g' \in \mathcal{V}'_\xi (g' \cup g'' = g_\eta)\}, \\ D^g_\xi &= \left\{\beta^\eta_\xi \mid \eta \in E_g\right\}. \end{aligned}$$

Since $|\mathcal{V}'_\xi| < F(\xi)$, we see that $|E_g| < F(\xi)$, hence $|D^g_\xi| < F(\xi)$. Clearly, if $g, \tilde{g} \in \mathcal{V}_\xi$ and $g \quad (Z_\xi \cap \mathcal{B}) = \tilde{g} \quad (Z_\xi \cap \mathcal{B})$, then $D^g_\xi = D^{\tilde{g}}_\xi$.

Let $\mathcal{A}_\xi$ be an antichain below $p_{\xi+1}$ such that $r \in \mathcal{A}_\xi$ implies $|r(\beta) \cap V^\beta_\xi| = 1$ for all $\beta \in Z_\xi$, and let $g_r \in \mathcal{V}_\xi$ be such that $g_r(\beta)$ is the single element of $r(\beta) \cap V^\beta_\xi$ for each $\beta \in Z_\xi$. We define

$$\dot{D}_\xi = \left\{(r, \check{D}^{g_r}_\xi) \mid r \in \mathcal{A}_\xi\right\}.$$

It is clear by the above that for each $r \in \mathcal{A}_\xi$ and η such that $g_r = g_\eta$ we have

$$r \quad \text{“}\, \dot{f}(\xi) = \check{\beta}^\eta_\xi \in \dot{D}^{g_r}_\xi \text{ and } |\dot{D}^{g_r}_\xi| < F(\xi)\,\text{”},$$

so by denseness

$$p_{\xi+1} \quad \text{“}\, \dot{f}(\xi) \in \dot{D}_\xi \text{ and } |\dot{D}_\xi| < F(\xi)\,\text{”}.$$

To see that $p_{\xi+1} \quad$ “$\dot{D}_\xi \in \mathbf{V}[\dot{G} \quad \mathcal{B}]$”, we argue within $\mathbf{V}[G \quad \mathcal{B}]$. For every $r, \tilde{r} \in \mathcal{A}_\xi$ such that both $r \quad \mathcal{B}$ and $\tilde{r} \quad \mathcal{B}$ are elements of $G \quad \mathcal{B}$ we see that the corresponding g_r and $g_{\tilde{r}}$ have the property that $g_r \quad (Z_\xi \cap \mathcal{B}) = g_{\tilde{r}} \quad (Z_\xi \cap \mathcal{B})$, and therefore $D^{g_r}_\xi = D^{g_{\tilde{r}}}_\xi$. Thus, we can fix any arbitrary such $r \in \mathcal{A}_\xi$ for which $r \quad \mathcal{B} \in G \quad \mathcal{B}$ holds, and see that

$$\mathbf{V}[G \quad \mathcal{B}] \quad \text{“}\, p_{\xi+1} \quad \mathcal{B}^{\mathrm{c}} \quad \dot{D}_\xi = \check{D}^{g_r}_\xi\,\text{”}.$$

Let $p' = \bigwedge_{\xi\in\kappa} p_\xi$ be the limit of the generalised fusion sequence, and let $\dot{\varphi}$ be a name such that $p' \quad$ “$\dot{\varphi} : \xi \mapsto \dot{D}_\xi$”, then $\dot{\varphi}$ names an F-slalom in $\mathbf{V}[G \quad \mathcal{B}]$ and $p' \quad$ “$\dot{f} \in^* \dot{\varphi}$”. □

If we let $\mathcal{B} = ?$ in the definition of the lemma, then we can simplify this lemma to the following corollary, providing us with the preservation of the F-Sacks property.

Corollary 5.2.4
If $\overline{\mathsf{ML}} = \prod^{\leq\kappa}_{\xi\in A} \mathsf{ML}^{h_\xi}_\kappa$ and each $h_\xi \leq^* h$ and $F : \alpha \mapsto (h(\alpha)^{|\alpha|})^+$, then $\overline{\mathsf{ML}}$ has the F-Sacks property.

Finally the following lemma is based on Theorem 5.1.8 and shows how we can use products of forcing notions $\mathsf{ML}^{h_\xi}_\kappa$ to increase the cardinality of $\mathfrak{d}^F_\kappa(\in^*)$. Although we know that each $\mathsf{ML}^{h_\xi}_\kappa$ adds an h_ξ-avoiding κ-real from Theorem 5.1.8, we will force with $\leq\kappa$-support product and hence do not add them successively, but side-by-side. We will need to argue with the chain condition of $\mathsf{ML}^{h_\xi}_\kappa$ to show that $\mathfrak{d}^F_\kappa(\in^*)$ will indeed be increased.

Lemma 5.2.5

Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of ordinals, and consider a sequence of functions $\langle h_\xi \mid \xi \in \mathcal{A}\rangle$. We define the $\leq\kappa$-support product $\overline{\mathsf{ML}} = \prod_{\xi\in\mathcal{A}}^{\leq\kappa} \mathsf{ML}_\kappa^{h_\xi}$ and we assume G is an $\overline{\mathsf{ML}}$-generic filter. Let $\langle S_\xi \mid \xi \in \mathcal{B}\rangle$ be a sequence of stationary sets. If F is such that $F(\alpha) \leq h_\xi(\alpha)$ for all $\alpha \in S_\xi$ and $\xi \in \mathcal{B}$, then $\mathbf{V}[G]$ "$|\mathcal{B}| \leq \mathsf{d}_\kappa^F(\in^*)$". ◁

Proof. The lemma is trivial if $|\mathcal{B}| \leq \kappa^+$, so we will assume that $|\mathcal{B}| \geq \kappa^{++}$.

We work in $\mathbf{V}[G]$. Let $\mu < |\mathcal{B}|$ and let $\{\varphi_\xi \mid \xi \in \mu\} \subseteq \mathrm{Loc}_\kappa^F$, then we want to describe some $f \in {}^\kappa\kappa$ such that $f \in^* \varphi_\xi$ for each $\xi \in \mu$. Since $\overline{\mathsf{ML}}$ is $<\kappa^{++}$-c.c., we could find $\mathcal{A}_\xi \subseteq \mathcal{A}$ with $|\mathcal{A}_\xi| \leq \kappa^+$ for each $\xi \in \mu$ such that $\varphi_\xi \in \mathbf{V}[G \quad \mathcal{A}_\xi]$. Since $|\mathcal{B}| > \mu \cdot \kappa^+$, we may fix some $\beta \in \mathcal{B} \setminus \bigcup_{\xi\in\mu} \mathcal{A}_\xi$ for the remainder of this proof. Let $f = \bigcap_{p\in G} p(\beta)$, then $f \in {}^\kappa\kappa$ is the generic κ-real added by the β-th term of the product $\overline{\mathsf{ML}}$.

Continuing the proof in the ground model, let $\dot f$ be an $\overline{\mathsf{ML}}$-name for f, let $\dot\varphi_\xi$ be an $\overline{\mathsf{ML}}$-name for φ_ξ, let $p \in \overline{\mathsf{ML}}$ and $\alpha_0 \in \kappa$. We want to find some $\alpha \geq \alpha_0$ and $q \leq p$ such that q "$\dot f(\alpha) \notin \dot\varphi_\xi(\alpha)$".

Let $C = \{\alpha \in \kappa \mid p(\beta) \cap {}^\alpha\kappa = \mathrm{Split}_\alpha(p(\beta))\}$, which is a club set by Lemma 5.1.7. Since S_β is stationary, there exists some $\alpha \geq \alpha_0$ such that $\alpha \in C \cap S_\beta$. Choose some $p_0 \leq p$ such that $p_0(\beta) = p(\beta)$ and such that there is a $Y \in [\kappa]^{<F(\alpha)}$ for which p_0 "$\dot\varphi_\xi(\alpha) = \check Y$". This is possible, since $\varphi_\xi \in \mathbf{V}[G \quad \mathcal{A}_\xi]$ and $\beta \notin \mathcal{A}_\xi$, therefore we could find $p_0' \in \overline{\mathsf{ML}} \quad \mathcal{A}_\xi$ with $p_0' \leq p \quad \mathcal{A}_\xi$ and Y with the aforementioned property, and then let $p_0(\eta) = p_0'(\eta)$ if $\eta \in \mathcal{A}_\xi$ and $p_0(\eta) = p(\eta)$ otherwise.

Each $t \in p_0(\beta) \cap {}^\alpha\kappa$ is an $h_\beta(\alpha)$-splitting node, hence the set $X = \{\chi \in \kappa \mid t^\frown\langle\chi\rangle \in p_0(\beta)\}$ has cardinality $|X| \geq h_\beta(\alpha)$. Because $\alpha \in S_\beta$ and $\beta \in B$, we have by our assumptions on F that $|Y| < F(\alpha) \leq h_\beta(\alpha) \leq |X|$. We can therefore find some $\chi \in X$ such that $\chi \notin Y$. Let $q \leq p_0$ be defined as

$$q(\eta) = \begin{cases} (p_0(\beta))_{t^\frown\langle\chi\rangle} & \text{if } \eta = \beta, \\ p_0(\eta) & \text{otherwise.} \end{cases}$$

Here $(p_0(\beta))_{t^\frown\langle\chi\rangle}$ is the subtree of $p_0(\beta)$ generated by the initial segment $t^\frown\langle\chi\rangle$. Then $q \leq p_0 \leq p$ and q "$\dot f(\alpha) \notin \check Y = \dot\varphi_\xi(\alpha)$". □

Lemma 5.2.6

Let $\mathcal{A}$ be a set of ordinals such that $\kappa < \mathrm{cf}(|\mathcal{A}|)$, let $\langle h_\xi \mid \xi \in \mathcal{A}\rangle$ be a sequence of functions, let $\overline{\mathsf{ML}} = \prod_{\xi\in\mathcal{A}}^{\leq\kappa} \mathsf{ML}_\kappa^{h_\xi}$ with $\overline{\mathsf{ML}}$-generic G, and let $F \in {}^\kappa\kappa$. Assuming $\mathbf{V}$ "$2^\kappa = \kappa^+$", it follows that $\mathbf{V}[G]$ "$2^\kappa = |\mathrm{Loc}_\kappa^F| = \kappa^+ \cdot |\mathcal{A}|$". ◁

Proof. This is a standard argument of counting names. □

We are now ready to use our product forcing to separate κ many cardinals of the form $\mathsf{d}_\kappa^h(\in^*)$.

Theorem 5.2.7

There exists a family of functions $\{g_\eta \mid \eta \in \kappa\} \subseteq {}^\kappa\kappa$ such that for any $\gamma \in \kappa^+$ and any increasing sequence $\langle \lambda_\xi \mid \xi \in \gamma \rangle$ of cardinals with $\kappa < \mathrm{cf}(\lambda_\xi)$ for all $\xi \in \gamma$ and any $\sigma : \kappa \to \gamma$, there exists a forcing extension in which $\mathsf{d}_\kappa^{g_\eta}(\in^*) = \lambda_{\sigma(\eta)}$ for all $\eta \in \kappa$. ◁

Proof. We assume that $\mathbf{V}$ “$2^\kappa = \kappa^+$”, or otherwise we first use a forcing to collapse 2^κ to become κ^+. By a result of Solovay (see e.g. [Jec03, Theorem 8.10]) there exists a family of κ many disjoint stationary subsets of κ, thus let $\{S_\eta \mid \eta \in \kappa\}$ be such a family. Let $\kappa \leq \gamma \in \kappa^+$ and $\sigma : \kappa \to \gamma$ be given. We will assume without loss of generality that σ is bijective, and hence that $\sigma^{-1} : \gamma \to \kappa$ is a well-defined bijection. Let $\langle \lambda_\xi \mid \xi \in \gamma \rangle$ be an increasing sequence of cardinals with $\mathrm{cf}(\lambda_\xi) > \kappa$ for all $\xi \in \gamma$.

Fix some $F \in {}^\kappa\kappa$ such that $F(\alpha)^{|\alpha|} = F(\alpha)$ and $2^{F(\alpha)} \leq F(\beta)$ for any $\alpha < \beta$. For each $\eta \in \kappa$ we define a function g_η as follows:

$$g_\eta(\alpha) = \begin{cases} (F(\alpha))^+ & \text{if } \alpha \in S_\eta, \\ (2^{F(\alpha)})^+ & \text{otherwise.} \end{cases}$$

For each $\xi \in \gamma$ we define $H_\xi \in {}^\kappa\kappa$ as follows:

$$H_\xi(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in \bigcup_{\zeta \in \xi} S_{\sigma^{-1}(\zeta)}, \\ 2^{F(\alpha)} & \text{otherwise.} \end{cases}$$

For each $\xi \in \gamma$ let $\mathcal{A}_\xi$ be a set of ordinals with $|\mathcal{A}_\xi| = \lambda_\xi$, such that $\langle \mathcal{A}_\xi \mid \xi \in \gamma \rangle$ is a sequence of mutually disjoint sets, and let $\mathcal{A} = \bigcup_{\xi \in \gamma} \mathcal{A}_\xi$. For each $\xi \in \gamma$ and $\beta \in \mathcal{A}_\xi$, we define $h_\beta = H_\xi$.

We now consider the product forcing $\overline{\mathsf{ML}} = \prod_{\beta \in \mathcal{A}}^{\leq\kappa} \mathsf{ML}_\kappa^{h_\beta}$. Let G be an $\overline{\mathsf{ML}}$-generic filter. We will fix some $\eta \in \kappa$, and let $\mathcal{B} = \bigcup_{\xi \in \sigma(\eta)+1} \mathcal{A}_\xi$ and $\mathcal{B}^c = \mathcal{A} \setminus \mathcal{B}$. By Lemma 5.2.6 we see that $(\mathrm{Loc}_\kappa^{g_\eta})^{\mathbf{V}[G \ \mathcal{B}]}$ has cardinality

$$\kappa^+ \cdot |\mathcal{B}| = \kappa^+ \cdot \left|\sup_{\xi \leq \sigma(\eta)} \mathcal{A}_\xi\right| = \kappa^+ \cdot \left|\mathcal{A}_{\sigma(\eta)}\right| = \lambda_{\sigma(\eta)}.$$

To use Lemma 5.2.3, we need that $h_\beta <^* g_\eta$ for all $\beta \in \mathcal{B}^c$, equivalently, that $H_\xi <^* g_\eta$ for all $\xi \in (\sigma(\eta), \gamma)$. But this is true for any $\xi \in (\sigma(\eta), \gamma)$, since $g_\eta(\alpha) = (F(\alpha))^+$ iff $\alpha \in S_\eta = S_{\sigma^{-1}(\sigma(\eta))}$ and because $\sigma(\eta) \in \xi$ we see that $H_\xi(\alpha) = F(\alpha)$. Meanwhile for all $\alpha \notin S_\eta$ we have $g_\eta(\alpha) = (2^{F(\alpha)})^+ > H_\xi(\alpha)$. Therefore Lemma 5.2.3 shows that $(\mathrm{Loc}_\kappa^{g_\eta})^{\mathbf{V}[G \ \mathcal{B}]}$ is a family in $\mathbf{V}[G]$ of size $\lambda_{\sigma(\eta)}$ that forms a witness for

$$\mathbf{V}[G] \quad \text{“}\mathsf{d}_\kappa^{g_\eta}(\in^*) \leq \lambda_{\sigma(\eta)}\text{”}.$$

On the other hand, if $\beta \in \mathcal{A}_{\sigma(\eta)}$, then $h_\beta = H_{\sigma(\eta)}$ and thus for any $\alpha \in S_\eta = S_{\sigma^{-1}(\sigma(\eta))}$ we see that $g_\eta(\alpha) = (F(\alpha))^+ \leq 2^{F(\alpha)} = H_{\sigma(\eta)}(\alpha)$. Therefore by Lemma 5.2.5 we see that

$$\mathbf{V}[G] \quad \text{“}\lambda_{\sigma(\eta)} = |\mathcal{A}_{\sigma(\eta)}| \leq \mathsf{d}_\kappa^{g_\eta}(\in^*)\text{”}.$$

In conclusion, we get for every $\eta \in \kappa$ that

$$\mathbf{V}[G] \quad \text{“}\lambda_{\sigma(\eta)} = \mathsf{d}_\kappa^{g_\eta}(\in^*)\text{”}.$$

□

Corollary 5.2.8
There exists functions h_ξ for each $\xi \in \kappa$ such that for any cardinals $\lambda_\xi > \kappa$ with $\mathrm{cf}(\lambda_\xi) > \kappa$ it is consistent that simultaneously $\mathsf{d}_\kappa^{h_\xi}(\in^*) = \lambda_\xi$ for all $\xi \in \kappa$.

5.3. κ^+ Many Localisation Cardinals

We saw in the previous section that we can use a partition of κ into disjoint stationary sets $\{S_\eta \mid \eta \in \kappa\}$, and associate a function g_η with each stationary S_η such that the cardinals $\mathsf{d}_\kappa^{g_\eta}(\in^*)$ can consistently be put in any arbitrary well-order.

It is natural to ask if we can do better than this, and separate κ^+ many cardinalities. Clearly we cannot do this using a disjoint family of stationary sets, since no such family of size κ^+ exists. Fortunately we can work around this by using an ***almost*** disjoint family of stationary sets, that is, a family $\mathcal{S}$ of stationary subsets of κ, such that $|S \cap S'| < \kappa$ for any distinct $S, S' \in \mathcal{S}$. Let us refer to such families as ***stationary almost disjoint families***, or ***sad families***.

The existence of a sad family of size 2^κ is a consequence of $\ _\kappa$. Let $\langle A_\alpha \mid \alpha \in \kappa\rangle$ be a $\ _\kappa$-sequence, that is, a sequence such that for any $X \in \mathcal{P}(\kappa)$ the following set is stationary:

$$S_X = \{\alpha \in \kappa \mid X \cap \alpha = A_\alpha\}.$$

If $X, Y \in \mathcal{P}(\kappa)$ are distinct, and ξ is the least element of the symmetric difference $X \triangle Y$, then it is easy to see that $S_X \cap S_Y \subseteq \xi + 1$, thus $\{S_X \mid X \in \mathcal{P}(\kappa)\}$ is a sad family of size 2^κ.

Let us first attempt to follow the reasoning from Theorem 5.2.7 to demonstrate the differences between using a family of disjoint stationary sets and a sad family as generators for our forcing parameters.

Assume $\mathbf{V}$ "$2^\kappa = \kappa^+$ and $\ _\kappa$" and fix a sad family $\{S_\eta \mid \eta \in \kappa^+\}$. We assume that $F \in {}^\kappa\kappa$ is some arbitrary function such that $F(\alpha)^{|\alpha|} = F(\alpha)$ and $2^{F(\alpha)} \leq F(\beta)$ for all $\alpha < \beta$. For every $\eta \in \kappa^+$ we can define the functions g_η, forming the parameters of the cardinal characteristics $\mathsf{d}_\kappa^{g_\eta}(\in^*)$ we wish to separate:

$$g_\eta(\alpha) = \begin{cases} (F(\alpha))^+ & \text{if } \alpha \in S_\eta, \\ (2^{F(\alpha)})^+ & \text{otherwise.} \end{cases}$$

In analogy with Theorem 5.2.7, we want to define functions H_ξ such that $\mathsf{ML}_\kappa^{H_\xi}$ keeps $\mathsf{d}_\kappa^{g_\eta}(\in^*)$ small when $\eta \in X$ and increases $\mathsf{d}_\kappa^{g_\eta}(\in^*)$ when $\eta \in Y$, where $\{X, Y\}$ forms a partition of κ^+. Assuming $H_\xi(\alpha) = H_\xi(\alpha)^{|\alpha|}$ for all $\alpha \in \kappa$, this means that we want to define H_ξ such that

when $\eta \in X$: $\quad H_\xi <^* g_\eta$,

when $\eta \in Y$: $\quad g_\eta(\alpha) \leq H_\xi(\alpha)$ for all $\alpha \in S$, where S is stationary.

Note that $g_\eta(\alpha)$ can only have two possible values, either $(F(\alpha))^+$ or $(2^{F(\alpha)})^+$, regardless of $\eta \in \kappa^+$. We can therefore assume without loss of generality that $H_\xi(\alpha)$ is either equal to $F(\alpha)$ or $2^{F(\alpha)}$, as in the construction from Theorem 5.2.7. Let z be the set on which H_ξ is small:

$$z = \{\alpha \in \kappa \mid H_\xi(\alpha) = F(\alpha)\}.$$

If $\eta \in X$, then $H_\xi(\alpha) < g_\eta(\alpha)$ is true for all $\alpha \notin S_\eta$, but since $H_\xi(\alpha) < g_\eta(\alpha)$ has to hold for almost all $\alpha \in \kappa$, we also need $|\{\alpha \in S_\eta \mid \alpha \notin z\}| < \kappa$. Let us fix the notation that $a \subseteq^* b$ iff $a \setminus c \subseteq b$ for some c with $|c| < \kappa$, then our condition above states that $S_\eta \subseteq^* z$ should hold.

On the other hand, if $\eta \in Y$, then $g_\eta(\alpha) \leq H_\xi(\alpha)$ is possible if

$$g_\eta(\alpha) = (F(\alpha))^+ \leq 2^{F(\alpha)} = H_\xi(\alpha).$$

Thus $(\kappa \setminus z) \cap S_\eta$ needs to be stationary. The assumption that $|z \cap S_\eta| < \kappa$ is sufficient for this.

Given our sad family $\mathcal{S} = \langle S_\eta \mid \eta \in \kappa^+ \rangle$, the existence of a set z such that $S_\eta \subseteq^* z$ for all $\eta \in X$ and $|z \cap S_\eta| < \kappa$ for all $\eta \in Y$, is not immediately clear. This forms the main obstacle in generalising Theorem 5.2.7. We can overcome this difficulty by adding a suitable z as described above generically through forcing.

We define the forcing $\mathsf{W}_{\mathcal{S}}^{X,Y}$ (where W stands for ***wedge***). If $s \in [\kappa]^{<\kappa}$, let σ_s be the least ordinal such that $s \subseteq \sigma_s$.

Definition 5.3.1

Given a sequence $\mathcal{S} = \langle S_\eta \mid \eta \in \kappa^+ \rangle$ of almost disjoint subsets of κ and a partition $\{X, Y\}$ of κ^+, we define $\mathsf{W}_{\mathcal{S}}^{X,Y}$ to have tuples $p = (s_p, A_p, B_p)$ as conditions, where $s_p \in [\kappa]^{<\kappa}$ and $A_p \in [X]^{<\kappa}$ and $B_p \in [Y]^{<\kappa}$ are such that

$$\bigcup_{\eta \in A_p} S_\eta \cap \bigcup_{\eta \in B_p} S_\eta \subseteq \sigma_{s_p}.$$

The ordering on $\mathsf{W}_{\mathcal{S}}^{X,Y}$ is given by $(s_q, A_q, B_q) \leq (s_p, A_p, B_p)$ if all of the following hold:

(i) $A_p \subseteq A_q$,
(ii) $B_p \subseteq B_q$,
(iii) $s_p = s_q \cap \sigma_{s_p}$,
(iv) $s_q \cap [\sigma_{s_p}, \sigma_{s_q}) \supseteq \bigcup_{\eta \in A_p} S_\eta \cap [\sigma_{s_p}, \sigma_{s_q})$,
(v) $s_q \cap \bigcup_{\eta \in B_p} S_\eta \subseteq \sigma_{s_p}$. ◁

If $G \subseteq \mathsf{W}_{\mathcal{S}}^{X,Y}$ is a generic filter, then let $z_G = \bigcup_{p \in G} s_p$. It is not hard to see that z_G indeed has the desired properties:

Lemma 5.3.2

If $\eta \in X$, then $S_\eta \subseteq^* z_G$. If $\eta \in Y$, then $|S_\eta \cap z_G| < \kappa$. ◁

Proof. Let $p \in \mathsf{W}_{\mathcal{S}}^{X,Y}$. It is clear from the way we have defined the forcing that for any $\eta \in A_p$ we have p "$\check{S}_\eta \subseteq^* \dot{z}_G$" and for any $\eta \in B_p$ we have p "$\check{S}_\eta \cap \dot{z}_G \subseteq \check{\sigma}_p \in \kappa$". Therefore, we are done if we prove that

1. for every $\eta \in X$ there is $q \leq p$ such that $\eta \in A_q$, and
2. for every $\eta \in Y$ there is $q \leq p$ such that $\eta \in B_q$.

Proving (1) and (2) happens in the same way, so we only prove (1) below.

Fix some $\eta \in X$. Since S_η is almost disjoint from S_ξ for all $\xi \in B_p$, we can define $\gamma_\xi \in \kappa$ such that $S_\xi \cap S_\eta \subseteq \gamma_\xi$ for each $\xi \in B_p$. Since $|B_p| < \kappa$ we see that $\bigcup_{\xi \in B_p} \gamma_\xi \in \kappa$. Pick some $\gamma \in \bigcup_{\xi \in A_p} S_\xi$ such that $\gamma \geq \sigma_{s_p} \cup \bigcup_{\xi \in B_p} \gamma_\xi$.

We define $q \leq p$ by

$$
\begin{aligned}
s_q &= s_p \cup \left(\bigcup_{\xi \in A_p} S_\xi \cap [\sigma_{s_p}, \gamma]\right), \\
A_q &= A_p \cup \{\eta\}, \\
B_q &= B_p.
\end{aligned}
$$

Note that $\gamma \in s_q$, thus $\sigma_{s_q} = \gamma + 1$. Furthermore, note that

$$
\begin{aligned}
&\bigcup_{\xi \in A_p} S_\xi \cap \bigcup_{\xi \in B_p} S_\xi \subseteq \sigma_{s_p} \leq \gamma \quad \text{and} \\
&S_\eta \cap \bigcup_{\xi \in B_p} S_\xi \subseteq \bigcup_{\xi \in B_p} \gamma_\xi \leq \gamma.
\end{aligned}
$$

Therefore, q is indeed a condition. □

We need to show that our forcing has several nice properties to satisfy our needs. Firstly, it is essential that the sad family $\{S_\eta \mid \eta \in \kappa^+\}$ will remain a sad family, in particular, the forcing should not destroy any stationary sets. Secondly, our forcing needs to preserve cardinals. In particular, we may not collapse κ^+ to κ, since our goal of proving the consistency of κ^+ many distinct cardinal characteristics requires our sad family to have cardinality κ^+. Thirdly, our forcing should preserve $2^\kappa = \kappa^+$, which we need for the forcing notions of type ML^h_κ afterwards.

All of these properties hold for $\mathsf{W}^{X,Y}_{\mathcal{S}}$ under the assumption that $|X| = \kappa$, since we can show that the forcing is $<\kappa$-closed and $(\kappa, <\kappa)$-centred in this case, and our forcing is small enough that it does not increase 2^κ.

Lemma 5.3.3
$\mathsf{W}^{X,Y}_{\mathcal{S}}$ is $<\kappa$-closed. ◁

Proof. Let $\gamma \in \kappa$ be limit and let $\langle p_\eta \mid \eta < \gamma \rangle$ be a descending sequence of conditions. We will write $p_\eta = (s_\eta, A_\eta, B_\eta)$. Let $p = (s_p, A_p, B_p)$ be given by $s_p = \bigcup_{\eta \in \gamma} s_\eta$ and $A_p = \bigcup_{\eta \in \gamma} A_\eta$ and $B_p = \bigcup_{\eta \in \gamma} B_\eta$. That p is a condition and that $p \leq p_\eta$ for each $\eta \in \gamma$ are easy to check. □

Corollary 5.3.4
$\mathsf{W}^{X,Y}_{\mathcal{S}}$ preserves stationary sets. ◁

Proof. See for example [Jec03, Lemma 23.7]. □

Lemma 5.3.5
If $|X| \leq \kappa$, then $\mathsf{W}^{X,Y}_{\mathcal{S}}$ is $(\kappa, <\kappa)$-centred. ◁

Proof. For any $s \in [\kappa]^{<\kappa}$ and $A \in [X]^{<\kappa}$ we define

$$W_{s,A} = \left\{ p \in \mathsf{W}_{\mathcal{S}}^{X,Y} \mid s_p = s \wedge A_p = A \right\}.$$

Since $|X| \leq \kappa$ implies that $|[\kappa]^{<\kappa} \times [X]^{<\kappa}| = \kappa$, we are done if we show that each $W_{s,A}$ is $<\kappa$-linked. Let $Q \in [W_{s,A}]^{<\kappa}$ and $B = \bigcup_{p \in Q} B_p$. We claim that $q = \langle s, A, B \rangle$ is a condition and that $q \leq p$ for all $p \in Q$.

Suppose that $\alpha \in \bigcup_{\eta \in A} S_\eta \cap \bigcup_{\eta \in B} S_\eta$, then there is $p \in Q$ such that $\alpha \in \bigcup_{\eta \in A} S_\eta \cap \bigcup_{\eta \in B_p} S_\eta$, and since p is a condition it follows that $\alpha \in \sigma_{s_p} = \sigma_s$. Hence q is a condition. To check that $q \leq p$ for each $p \in Q$, note that (i), (ii) and (iii) of Definition 5.3.1 are immediate, while (iv) and (v) hold vacuously by $s_p = s_q$. □

Corollary 5.3.6
If $|X| \leq \kappa$, then $\mathsf{W}_{\mathcal{S}}^{X,Y}$ preserves all cardinalities. ◁

Finally, we have to look at adding multiple generics of forcing notions of the type $\mathsf{W}_{\mathcal{S}}^{X,Y}$. Our goal is to define functions H_ξ for each $\xi \in \kappa^+$. Fix some bijection $\sigma : \kappa^+ \to \kappa^+$, then we want to add a generic set z for the forcing $\mathsf{W}_{\mathcal{S}}^{X_\xi, Y_\xi}$ for each $\xi \in \kappa^+$, where $X_\xi = \sigma(\xi)$ and $Y_\xi = \kappa^+ \setminus X_\xi$. This means that we also need to guarantee that a $<\kappa$-support product of size κ^+ of forcing notions of the form $\mathsf{W}_{\mathcal{S}}^{X,Y}$ behaves nicely, in the sense that it preserves cardinals, stationary sets and $2^\kappa = \kappa^+$.

Lemma 5.3.7
Let $\langle \{X_\xi, Y_\xi\} \mid \xi \in \kappa^+ \rangle$ be a sequence of partitions of κ^+ such that $|X_\xi| \leq \kappa$ for each $\xi \in \kappa^+$ and let $\overline{\mathsf{W}} = \prod_{\xi \in \kappa^+}^{\leq \kappa} \mathsf{W}_{\mathcal{S}}^{X_\xi, Y_\xi}$. Then $\overline{\mathsf{W}}$ is $<\kappa$-closed, $<\kappa^+$-c.c. and if G is $\overline{\mathsf{W}}$-generic over $\mathbf{V}$ and $\mathbf{V}$ “$2^\kappa = \kappa^+$”, then $\mathbf{V}[G]$ “$2^\kappa = \kappa^+$”. ◁

Proof. Note that each term $\mathsf{W}_{\mathcal{S}}^{X_\xi, Y_\xi}$ is $<\kappa$-closed, thus $\overline{\mathsf{W}}$ is also $<\kappa$-closed by Theorem 4.1.15.

That $\overline{\mathsf{W}}$ is $<\kappa^+$-c.c. is proved using a Δ-system argument similar to Theorem 4.1.17 and Lemma 4.1.20.

That $2^\kappa = \kappa^+$ will remain true, follows from an argument by counting names, using that $|\mathsf{W}_{\mathcal{S}}^{X_\xi, Y_\xi}| = \kappa^+$ for each $\xi \in \kappa^+$, and that the product has κ^+ many terms, thus $|\overline{\mathsf{W}}| = \kappa^+$. □

Corollary 5.3.8
$\overline{\mathsf{W}}$ preserves cardinals and stationary sets. ◁

Now we are finally ready to prove our last theorem, which extends Theorem 5.2.7, and shows that there can be consistently κ^+ many distinct cardinal characteristics of the form $\mathfrak{d}_\kappa^h(\in^*)$.

Theorem 5.3.9
Assuming $2^\kappa = \kappa^+$ and $_\kappa$, there exists a family of functions $\{g_\eta \mid \eta \in \kappa^+\} \subseteq {}^\kappa\kappa$ such that for any increasing sequence $\langle \lambda_\xi \mid \xi \in \kappa^+ \rangle$ of cardinals with $\kappa < \mathrm{cf}(\lambda_\xi)$ and any function $\sigma : \kappa^+ \to \kappa^+$, there exists a forcing extension in which $\mathfrak{d}_\kappa^{g_\eta}(\in^*) = \lambda_{\sigma(\eta)}$ for all $\eta \in \kappa^+$. ◁

Proof. We start with a model $\mathbf{V}$ “$2^\kappa = \kappa^+$ and ${}_\kappa$” containing a sad family $\mathcal{S} = \langle S_\eta \mid \eta \in \kappa^+\rangle$, and we will assume without loss of generality that $\sigma : \kappa^+ \to \kappa^+$ is a bijection. We define the functions g_η for each $\eta \in \kappa^+$ as

$$g_\eta(\alpha) = \begin{cases} (F(\alpha))^+ & \text{if } \alpha \in S_\eta, \\ (2^{F(\alpha)})^+ & \text{otherwise.} \end{cases}$$

For each $\eta \in \kappa^+$ we define the partition $\{X_\eta, Y_\eta\}$ of κ^+ by

$$X_\eta = \sigma^{-1}[\sigma(\eta)] = \big\{\zeta \in \kappa^+ \mid \sigma(\zeta) \in \sigma(\eta)\big\} \quad \text{and}$$
$$Y_\eta = \kappa^+ \setminus X_\eta.$$

We then force with a $<\kappa$-support product $\overline{\mathsf{W}} = \prod^{\leq\kappa}_{\eta\in\kappa^+} \mathsf{W}_{\mathcal{S}}^{X_\eta, Y_\eta}$. Note in particular that $|X_\eta| \leq \kappa$. Let G be $\overline{\mathsf{W}}$-generic, then we will work in $\mathbf{V}[G]$. We define $z_{\sigma(\eta)} = \bigcup_{p\in G} s_{p(\eta)}$, then $z_{\sigma(\eta)}$ is $\mathsf{W}_{\mathcal{S}}^{X_\eta,Y_\eta}$-generic over $\mathbf{V}$.

By Lemma 5.3.7, we know that $\mathbf{V}[G]$ “$2^\kappa = \kappa^+ + \check{\mathcal{S}}$ is a sad family”. Moreover, given $\eta \in \kappa^+$ we know by Lemma 5.3.2 that $S_\zeta \subseteq^* z_{\sigma(\eta)}$ for all $\zeta \in X_\eta$ and $|S_\zeta \cap z_{\sigma(\eta)}| < \kappa$ for all $\zeta \in Y_\eta$. Equivalently, using the definition of X_η and Y_η, if $\xi \in \kappa^+$, then we have $S_\zeta \subseteq^* z_\xi$ for all $\zeta \in \kappa^+$ such that $\sigma(\zeta) \in \xi$ and $|S_\zeta \cap z_\xi| < \kappa$ for all $\zeta \in \kappa^+$ such that $\sigma(\zeta) \in [\xi, \kappa^+)$.

For each $\xi \in \kappa^+$ we define $H_\xi \in {}^\kappa\kappa$ as follows:

$$H_\xi(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in z_\xi, \\ 2^{F(\alpha)} & \text{otherwise.} \end{cases}$$

The remainder of the proof mirrors the proof of Theorem 5.2.7 almost exactly.

For each $\xi \in \kappa^+$ let $\mathcal{A}_\xi$ be a set of ordinals with $|\mathcal{A}_\xi| = \lambda_\xi$, such that $\langle \mathcal{A}_\xi \mid \xi \in \kappa^+\rangle$ is a sequence of mutually disjoint sets, and let $\mathcal{A} = \bigcup_{\xi\in\kappa^+} \mathcal{A}_\xi$. For $\xi \in \kappa^+$ and $\beta \in \mathcal{A}_\xi$, we define $h_\beta = H_\xi$.

We now consider the $\leq\kappa$-support product $\overline{\mathsf{ML}} = \prod^{\leq\kappa}_{\beta\in\mathcal{A}} \mathsf{ML}_\kappa^{h_\beta}$. Let K be $\overline{\mathsf{ML}}$-generic. We will fix some $\eta \in \kappa^+$, and let $\mathcal{B} = \bigcup_{\xi\in\sigma(\eta)+1} \mathcal{A}_\xi$ and $\mathcal{B}^{\mathrm{c}} = \mathcal{A} \setminus \mathcal{B}$. By Lemma 5.2.6 we see that $(\mathrm{Loc}_\kappa^{g_\eta})^{\mathbf{V}[G][K\ \mathcal{B}]}$ has cardinality

$$\kappa^+ \cdot |\mathcal{B}| = \kappa^+ \cdot \Big|\sup\nolimits_{\xi\leq\sigma(\eta)} \mathcal{A}_\xi\Big| = \kappa^+ \cdot \big|\mathcal{A}_{\sigma(\eta)}\big| = \lambda_{\sigma(\eta)}.$$

To use Lemma 5.2.3, we need that $h_\beta <^* g_\eta$ for all $\beta \in \mathcal{B}^{\mathrm{c}}$, equivalently, that $H_\xi <^* g_\eta$ for all $\xi \in (\sigma(\eta), \kappa^+)$. But this is true for any $\xi \in (\sigma(\eta), \kappa^+)$, since $g_\eta(\alpha) = (F(\alpha))^+$ iff $\alpha \in S_\eta$, while $H_\xi(\alpha) = F(\alpha)$ iff $\alpha \in z_\xi$, and because $\sigma(\eta) \in \xi$ we have $S_\eta \subseteq^* z_\xi$. Therefore Lemma 5.2.3 shows that $(\mathrm{Loc}_\kappa^{g_\eta})^{\mathbf{V}[G][K\ \mathcal{B}]}$ is a family in $\mathbf{V}[G][K]$ of size $\lambda_{\sigma(\eta)}$ that forms a witness for

$$\mathbf{V}[G][K] \quad \text{“}\,\mathfrak{d}_\kappa^{g_\eta}(\in^*) \leq \lambda_{\sigma(\eta)}\text{”}.$$

On the other hand, if $\beta \in \mathcal{A}_{\sigma(\eta)}$, then $h_\beta = H_{\sigma(\eta)}$ and thus $\sigma(\eta) \in [\sigma(\eta), \kappa^+)$ implies that $|S_\eta \cap z_{\sigma(\eta)}| < \kappa$. In particular, $S_\eta \setminus z_{\sigma(\eta)}$ is stationary and if $\alpha \in S_\eta \setminus z_{\sigma(\eta)}$, then $g_\eta(\alpha) \leq H_{\sigma(\eta)}(\alpha)$. Hence by Lemma 5.2.5 we see that

$$\mathbf{V}[G][K] \quad \text{“}\,\lambda_{\sigma(\eta)} = |\mathcal{A}_{\sigma(\eta)}| \leq \mathfrak{d}_\kappa^{g_\eta}(\in^*)\text{”}.$$

In conclusion, we get for every $\eta \in \kappa$ that

$$\mathbf{V}[G][K] \Vdash \text{“} \lambda_{\sigma(\eta)} = \mathfrak{d}^{g_\eta}_\kappa(\in^*) \text{”}. \qquad \square$$

5.4. Open Questions

With Theorem 5.3.9 we improved the known consistency of $\mathfrak{d}^{\mathrm{pow}}_\kappa(\in^*) < \mathfrak{d}^{\mathrm{id}}_\kappa(\in^*)$ to a family of κ^+ many cardinal characteristics that are mutually independent in the sense that any ordering of the cardinals with order-type κ^+ is consistent. This answers Questions 72 and 73 from [BBTFM18] positively. Moreover, we have shown that there exist functions $h, h' \in {}^\kappa\kappa$ for which it is consistent that $\mathfrak{d}^h_\kappa(\in^*) < \mathfrak{d}^{h'}_\kappa(\in^*)$, but also that it is consistent that $\mathfrak{d}^{h'}_\kappa(\in^*) < \mathfrak{d}^h_\kappa(\in^*)$.

It is natural to ask if we can do better than this:

Question 5.4.1
Is it consistent that there exists a family of functions $\{h_\xi \mid \xi \in \kappa^{++}\}$ such that each $\mathfrak{d}^{h_\xi}_\kappa(\in^*)$ has a distinct value? Is it consistent that there is a model with 2^κ many distinct values for cardinals of the form $\mathfrak{d}^h_\kappa(\in^*)$? ◁

Our method of separating cardinals uses a forcing ML^h_κ that requires $2^\kappa = \kappa^+$ in the ground model to have the $<\kappa^{++}$-c.c., hence if we start with a family of functions of size κ^{++}, our forcing may collapse κ^{++}. This makes it hard to answer the above question using the method presented in this chapter.

Another limitation of our method, is that we restrict our attention to forcing notions that have the F-Sacks properties where $F(\alpha) = F(\alpha)^{|\alpha|}$. Essentially, we know how to separate cardinals with a parameter h from cardinals with a parameter 2^h, and thus we make a jump on the order of a power set operation. It is unclear whether a finer structure can be discovered between these cardinals, motivating the following question:

Question 5.4.2
Is it consistent that there exist $h_0, h_1, h_2 \in {}^\kappa\kappa$ such that $|h_0(\alpha)| < |h_1(\alpha)| < 2^{|h_0(\alpha)|} = h_2(\alpha)$ and $\mathfrak{d}^{h_2}_\kappa(\in^*) < \mathfrak{d}^{h_1}_\kappa(\in^*) < \mathfrak{d}^{h_0}_\kappa(\in^*)$? ◁

The localisation cardinals $\mathfrak{d}^h_\kappa(\in^*)$ have their natural dual in the avoidance cardinals $\mathfrak{b}^h_\kappa(\in^*)$ defined in Section 2.4. In general, for many cardinal characteristics $\mathfrak{x}, \mathfrak{y}$ with duals $\mathfrak{x}', \mathfrak{y}'$ it is the case that if $\mathfrak{x} < \mathfrak{y}$ is consistent, then $\mathfrak{y}' < \mathfrak{x}'$ is consistent as well. This motivates the following question, which has also been asked as Question 71 from [BBTFM18]:

Question 5.4.3
Do there exist functions h, h' such that $\mathfrak{b}^h_\kappa(\in^*) < \mathfrak{b}^{h'}_\kappa(\in^*)$ is consistent? ◁

One candidate for a forcing notion would be κ-localisation forcing, if we can answer Question 4.5.3 positively. Other candidates could be perfect tree forcing notions whose nodes are splitting on almost every level (e.g. κ-Laver trees), but such forcing notions behave quite differently from the classical case. Even if such forcing notions happen to have the right properties, the last obstacle is preservation of such properties.

6

Quite a Few Antiavoidance Cardinals

In the previous chapter we showed the consistency of κ^+ many different localisation numbers $\mathfrak{d}^h_\kappa(\in^*)$. With the same forcing techniques one could also separate κ^+ many bounded localisation numbers $\mathfrak{d}^{b,h}_\kappa(\in^*)$, as per the comment at the end of Section 5.1. The antiavoidance numbers $\mathfrak{d}^h_\kappa(\ni^\infty)$ cannot be separated, since these are equal to $\mathrm{cov}(\mathcal{M}_\kappa)$ for any $h \in {}^\kappa\kappa$ by Corollary 3.3.8. In this chapter we consider whether it is possible to separate multiple bounded antiavoidance numbers $\mathfrak{d}^{b,h}_\kappa(\ni^\infty)$. Before we do so, let us give a brief history[1] of the consistency of separating (anti)localisation and (anti)avoidance numbers defined on bounded ***classical*** Baire spaces (that is, on $\prod b$ with $b \in {}^\omega\omega$).

Goldstern & Shelah [GS93] proved the consistency of $\aleph_1$ many localisation numbers $\mathfrak{d}^{b,h}(\in^*)$ of different cardinality, which was later improved by Kellner [Kel08] to show the consistency of $2^{\aleph_0}$ many different localisation numbers $\mathfrak{d}^{b,h}(\in^*)$, and together with Shelah [KS09, KS12] this was extended further to the consistency of continuum many antilocalisation numbers $\mathfrak{b}^{b,h}(\ni^\infty)$ of different cardinality. Kellner & Shelah's method uses a form of creature forcing that resembles forcing with a countable support product of proper forcing notions. Later work by Kamo & Osuga [KO14] showed how antilocalisation is related to a family of parametrised ideals, known as Yorioka ideals[2], and used this to give a different forcing construction of continuum many different antilocalisation numbers $\mathfrak{b}^{b,h}(\ni^\infty)$ under the assumption of the existence of an inaccessible, using a finite support iteration of c.c.c. forcing notions. Combining techniques developed by Brendle & Mejía [BM14] with those from Kamo & Osuga, it was shown by Cardona & Mejía [CM19] that consistently there exist continuum many different antiavoidance numbers $\mathfrak{d}^{b,h}(\ni^\infty)$, again assuming the existence of an inaccessible. Klausner & Mejía [KM22] then showed that consistently there are uncountably many different localisation numbers $\mathfrak{d}^{b,h}(\in^*)$ as well as antiavoidance numbers $\mathfrak{d}^{b,h}(\ni^\infty)$ and finally Cardona, Klausner & Mejía [CKM21] showed the consistency of continuum many different localisation, antilocalisation, avoidance and antiavoidance cardinals without the use of an inaccessible cardinal.

Our goal is to show the consistency of 2^κ many different (anti)localisation and (anti)avoidance numbers of the higher Baire space ${}^\kappa\kappa$, possibly by mimicking the techniques used in the classical proofs. Apart from the results from the previous chapter, this chapter will form a generalisation of the forcing construction from [KM22], although our conclusion will be significantly weaker. We will show that if κ is inaccessible, then there exist κ many functions b_α, h_α such that for any finite $A \subseteq \kappa$ it is consistent that $\mathfrak{d}^{b_\alpha,h_\alpha}_\kappa(\ni^\infty)$ are mutually distinct for all $\alpha \in A$.

We will increase a cardinal of the form $\mathfrak{d}^{b,h}_\kappa(\ni^\infty)$ by generically adding a (b,h)-antilocalising slalom. At the same time, we will make sure that the forcing notion we use preserves $\mathfrak{d}^{b',h'}_\kappa(\ni^\infty)$

[1] A more detailed version of this history could be found in [CKM21, Section 1].

[2] Yorioka ideals were first described by Yorioka [Yor02] to study the strong measure zero ideal.

for some other parameters b', h'. As with our construction from the previous section, we will consider a forcing notion consisting of higher perfect trees. In this case we use trees on $\mathrm{Loc}^{b,h}_{<\kappa}$ instead of trees on ${}^{<\kappa}\kappa$, so that our generic object will be the desired (b,h)-antilocalising slalom.

Since our proof follows the construction from [KM22] in large lines, we will give reference to the corresponding classical lemmas where appropriate. One main difference between our forcing notion and the forcing notion described in [KM22], is that we will not work with uniform trees. That is, the forcing notion of [KM22] is comparable to Silver forcing, and has partial functions as conditions. Our forcing notion is comparable to Sacks or Miller forcing. We made the choice to use a non-uniform forcing notion to make a better comparison with the forcing notion from Chapter 5. Both forcing notions with uniform perfect trees and with nonuniform perfect trees will have the same effect on $\mathfrak{d}^{b,h}_{\kappa}(\ni^{\infty})$, thus this change is not significant for the results we will prove.

Nota Bene! We will assume in this chapter without mention that κ is inaccessible and that b, h are increasing cofinal cardinal functions. This also extends to indexed or accented functions using the symbols b, h, such as b_ξ, h', and so on.

6.1. The Forcing Notion $\mathbb{Q}^{b,h}_{\kappa}$

In this section we will define a forcing notion $\mathbb{Q}^{b,h}_{\kappa}$ that consists of closed perfect trees on $\mathrm{Loc}^{b,h}_{\kappa}$ and will add a generic (b,h)-slalom. One should compare $\mathbb{Q}^{b,h}_{\kappa}$ to κ-Miller Lite forcing from Definition 5.1.1. We will first define a norm on subsets of $[b(\alpha)]^{<h(\alpha)}$.

Definition 6.1.1
Given $M \subseteq [b(\alpha)]^{<h(\alpha)}$ let $\|M\|_{b,\alpha}$ be the least cardinal $\lambda \in \kappa$ for which there exists $y \in [b(\alpha)]^{\lambda}$ such that for all $x \in M$ we have $y \nsubseteq x$, i.e., the least size of a subset y of $b(\alpha)$ such that no superset of y is contained in M. $\triangleleft$

Definition 6.1.2 — *cf. [KM22, Definition 3.1] for* ${}^{\omega}\omega$
We define a forcing notion $\mathbb{Q}^{b,h}_{\kappa}$ where conditions $T \in \mathbb{Q}^{b,h}_{\kappa}$ are trees $T \subseteq \mathrm{Loc}^{b,h}_{<\kappa}$ such that

(i) T is perfect and closed under splitting (see Definition 4.4.1),
(ii) if $u \in \mathrm{Split}_\alpha(T)$, then $\|\mathrm{suc}(u,T)\|_{b,\mathrm{dom}(u)} \geq |\alpha|$,
(iii) for any $u \in T$, if $\|\mathrm{suc}(u,T)\|_{b,\mathrm{dom}(u)} < 2$, then u is non-splitting.

ordered by $T \leq S$ iff $T \subseteq S$ and for $u \in T$ we have $\|\mathrm{suc}(u,T)\|_{b,\mathrm{dom}(u)} < \|\mathrm{suc}(u,S)\|_{b,\mathrm{dom}(u)}$ whenever $\mathrm{suc}(u,T) \neq \mathrm{suc}(u,S)$. $\triangleleft$

Here (iii) is necessary to ensure that the intersection of all trees in a generic filter forms a branch. If we allow $\|\mathrm{suc}(u,T)\|_{b,\alpha} = 1$ without $\mathrm{suc}(u,T)$ being a singleton, then we have no way of decreasing the norm of $\mathrm{suc}(u,T)$ any further.

For $A \subseteq \mathbf{Ord}$ and $T \in \mathbb{Q}^{b,h}_{\kappa}$, we define a ***collapse*** of $T \in \mathbb{Q}^{b,h}_{\kappa}$ on A as a condition $T' \leq T$ such that u is non-splitting in T' for all $u \in \mathrm{Lev}_\alpha(T')$ with $\alpha \in A$ and $\mathrm{suc}(u,T') = \mathrm{suc}(u,T)$ for all

$u \in \mathrm{Lev}_\alpha(T')$ with $\alpha \in \kappa \setminus A$. It is clear from the definition of the forcing notion that such a collapse exists for any set A that is the complement of a club set.

It is also clear that $\mathsf{Q}^{b,h}_\kappa$ adds a generic element to $\mathrm{Loc}^{b,h}_\kappa$, in the sense that if $G \subseteq \mathsf{Q}^{b,h}_\kappa$ is a generic filter over $\mathbf{V}$, then $\varphi_G = \bigcap G \in \mathrm{Loc}^{b,h}_\kappa$ and $\mathbf{V}[G] = \mathbf{V}[\varphi_G]$.

Definition 6.1.3 — *cf. [KM22, Observation 3.2] for* ${}^\omega\omega$

Define $(\mathsf{Q}^{b,h}_\kappa)^* \subseteq \mathsf{Q}^{b,h}_\kappa$ as the set of all $T \in \mathsf{Q}^{b,h}_\kappa$ such that for each $\alpha \in \kappa$ there exists $s_\alpha(T) \in \kappa$ such that $\mathrm{Split}_\alpha(T) = \mathrm{Lev}_{s_\alpha(T)}(T)$ and $\|\mathrm{suc}(u,T)\|_{b,s_\alpha(T)} \geq |s_\alpha(T)|$ for all $u \in \mathrm{Split}_\alpha(T)$. ◁

We will fix the notation $s_\alpha(T)$ to be as in the above definition for any $T \in (\mathsf{Q}^{b,h^*}_\kappa)$ and $\alpha \in \kappa$.

Lemma 6.1.4

$(\mathsf{Q}^{b,h}_\kappa)^*$ densely embeds into $\mathsf{Q}^{b,h}_\kappa$. ◁

Proof. Let $T \in \mathsf{Q}^{b,h}_\kappa$. Given $\alpha = \alpha_0 \in \kappa$, let

$$\alpha_{n+1} = \sup\left\{\mathrm{dom}(u) \;\middle|\; u \in \mathrm{Split}_{\alpha_n}(T)\right\}.$$

Note that $\langle \alpha_n \mid n \in \omega\rangle$ is increasing. Let $C = \{\sup_{n\in\omega} \alpha_n \mid \alpha \in \kappa\}$, then C is easily seen to be club. Note that if $\xi \in C$ and $u \in \mathrm{Split}_\xi(T)$, then $\mathrm{dom}(u) = \xi$. By (ii) of Definition 6.1.2 we see that $|\mathrm{dom}(u)| \leq \|\mathrm{suc}(u,T)\|_{b,\mathrm{dom}(u)}$ for all $u \in \mathrm{Split}_\xi(T)$ with $\xi \in C$.

Finally, let $T^* \leq T$ be a collapse of T on $\kappa \setminus C$, then $T^* \in (\mathsf{Q}^{b,h}_\kappa)^*$. □

Lemma 6.1.5

$\mathsf{Q}^{b,h}_\kappa$ is $<\kappa$-closed and $<(2^\kappa)^+$-c.c. ◁

Proof. Let $\lambda < \kappa$ and $\langle T_\xi \in \mathsf{Q}^{b,h}_\kappa \mid \xi \in \lambda\rangle$ be a descending sequence of conditions, then $\bigcap_{\xi\in\lambda} T_\xi$ is a condition below all T_ξ. The key observation in proving $<\kappa$-closure is the following claim: if $u \in T = \bigcap_{\xi\in\lambda} T_\xi$, then there is $\eta \in \lambda$ such that $\mathrm{suc}(u,T) = \mathrm{suc}(u,T_\eta)$. The remainder follows as in Lemma 5.1.2.

Suppose that $u \in T$ and let $\lambda_\xi = \|\mathrm{suc}(u,T_\xi)\|_{b,\mathrm{dom}(u)}$, then the ordering on $\mathsf{Q}^{b,h}_\kappa$ dictates that $\langle \lambda_\xi \mid \xi \in \lambda\rangle$ is a descending sequence of cardinals, hence there is $\eta \in \lambda$ such that $\lambda_\xi = \lambda_\eta$ for all $\xi \in [\eta,\lambda)$. But then $\mathrm{suc}(u,T_\xi) = \mathrm{suc}(u,T_\eta)$ for all $\xi \in [\eta,\lambda)$ by the ordering on $\mathsf{Q}^{b,h}_\kappa$.

That $\mathsf{Q}^{b,h}_\kappa$ is $<(2^\kappa)^+$-c.c. is immediate by $|\mathsf{Q}^{b,h}_\kappa| = 2^\kappa$. □

As a corollary to the above lemma, we see that $\mathsf{Q}^{b,h}_\kappa$ preserves all cardinalities $\leq\kappa$ and $>2^\kappa$. We will prove the preservation of κ^+ later, after Lemma 6.1.10, thus actually we see that all cardinalities are preserved if we assume $2^\kappa = \kappa^+$ in the ground model.

Define a fusion ordering $\langle \leq_\alpha \mid \alpha \in \kappa\rangle$ as in Definition 4.4.5 by $T \leq_\alpha S$ iff $T \leq S$ and $\mathrm{Split}_\alpha(T) = \mathrm{Split}_\alpha(S)$. It is easy to see that this is a fusion ordering.

Lemma 6.1.6 — *cf. [KM22, Lemma 3.5(b)] for* ${}^\omega\omega$

$\mathsf{Q}^{b,h}_\kappa$ is closed under fusion, that is, if $\langle T_\alpha \mid \alpha \in \kappa\rangle$ is a sequence in $\mathsf{Q}^{b,h}_\kappa$ such that $T_\beta \leq_\alpha T_\alpha$ for any $\beta > \alpha$, then $T = \bigcap_{\alpha\in\kappa} T_\alpha \in \mathsf{Q}^{b,h}_\kappa$ and $T \leq_\alpha T_\alpha$ for all $\alpha \in \kappa$. ◁

Proof. Similar to Lemmas 4.4.6 and 5.1.4. □

In the above lemma, we will note that if $T_\alpha \in (\mathsf{Q}_\kappa^{b,h})^*$ for all α, then $T \in (\mathsf{Q}_\kappa^{b,h})^*$ as well.

Lemma 6.1.7 — *cf. [KM22, Lemma 3.4] for* ${}^\omega\omega$

If $\alpha \in \kappa$, $T \in \mathsf{Q}_\kappa^{b,h}$ and $\mathcal{D} \subseteq \mathsf{Q}_\kappa^{b,h}$ is open dense, then there exists $T' \leq_\alpha T$ such that for any $v \in \mathrm{Split}_{\alpha+1}(T')$ we have $(T')_v \in \mathcal{D}$. Furthermore, if $T \in (\mathsf{Q}_\kappa^{b,h})^*$, then we can also find $T' \in (\mathsf{Q}_\kappa^{b,h})^*$ satisfying the above. ◁

Proof. Enumerate $\mathrm{Split}_{\alpha+1}(T)$ as $\langle v_\xi \mid \xi \in \mu\rangle$ and pick some $T_\xi \leq (T)_{v_\xi} \cap \mathcal{D}$ for each $\xi \in \mu$ with $T_\xi \in (\mathsf{Q}_\kappa^{b,h})^*$, then $T' = \bigcup_{\xi\in\mu} T_\xi$ satisfies the above. Note that by construction $\mathrm{Split}_{\alpha+1}(T) \subseteq T'$. It is easy to see that if $T \in (\mathsf{Q}_\kappa^{b,h})^*$, then $T' \in (\mathsf{Q}_\kappa^{b,h})^*$ holds as well. □

The following lemma shows that $\mathsf{Q}_\kappa^{b,h}$ satisfies a generalisation of Baumgartner's Axiom A to the context of ${}^\kappa\kappa$.

Lemma 6.1.8 — *cf. [KM22, Lemma 3.5(c)] for* ${}^\omega\omega$

If $A \subseteq \mathsf{Q}_\kappa^{b,h}$ is an antichain, $T \in \mathsf{Q}_\kappa^{b,h}$ and $\alpha \in \kappa$, then there is a condition $T' \leq_\alpha T$ in $\mathsf{Q}_\kappa^{b,h}$ such that T' is compatible with less than κ elements of A. ◁

Proof. Let $\mathcal{D}$ be the set of $S \in \mathsf{Q}_\kappa^{b,h}$ such that $S \leq R$ for some $R \in A$ or such that S is incompatible with all elements of A, then $\mathcal{D}$ is open dense. By Lemma 6.1.7 there is $T' \in \mathsf{Q}_\kappa^{b,h}$ with $T' \leq_\alpha T$ such that $(T')_v \in \mathcal{D}$ for all $v \in \mathrm{Split}_{\alpha+1}(T')$. Note that $|\mathrm{Split}_{\alpha+1}(T')| < \kappa$ and that $R \in A$ is compatible with T' iff there exists $v \in \mathrm{Split}_{\alpha+1}(T')$ such that $(T')_v \leq R$. It follows that less than κ many elements of A are compatible with T'. □

We will also prove that $\mathsf{Q}_\kappa^{b,h}$ has continuous reading of names. In fact, we will prove a different property that implies continuous reading, which is referred to as *early reading of names* in [KM22]. The preservation of κ^+ is a straightforward consequence of this property.

Definition 6.1.9 — *cf. [KM22, Definition 3.6] for* ${}^\omega\omega$

Let $T \in \mathsf{Q}_\kappa^{b,h}$ and $\dot\tau$ be a $\mathsf{Q}_\kappa^{b,h}$-name such that T "$\dot\tau : \kappa \to \mathbf{V}$". Then we say that T *reads* $\dot\tau$ *early* if $(T)_u$ decides $\dot\tau \quad \alpha$ for every $\alpha \in \kappa$ and $u \in \mathrm{Lev}_\alpha(T)$. ◁

Lemma 6.1.10 — *cf. [KM22, Lemmas 3.7 and 3.12] for* ${}^\omega\omega$

Let $T \in \mathsf{Q}_\kappa^{b,h}$ and $\dot\tau$ a $\mathsf{Q}_\kappa^{b,h}$-name such that T "$\dot\tau : \kappa \to \mathbf{V}$". Then there exists $T' \leq T$ with $T' \in (\mathsf{Q}_\kappa^{b,h})^*$ such that T' reads $\dot\tau$ early. ◁

Proof. Let $\mathcal{D}_\alpha = \{T \in \mathsf{Q}_\kappa^{b,h} \mid T \text{ decides } \dot\tau \quad \alpha\}$ and note that $\mathcal{D}_\alpha$ is open dense for each $\alpha \in \kappa$. We will construct a fusion sequence.

Let $T_0 \leq T$ be such that $T_0 \in (\mathsf{Q}_\kappa^{b,h})^*$. Given T_α, use Lemma 6.1.7 to define $T_{\alpha+1} \in (\mathsf{Q}_\kappa^{b,h})^*$ such that $T_{\alpha+1} \leq_\alpha T_\alpha$ and for any $v \in \mathrm{Split}_{\alpha+1}(T_{\alpha+1})$ we have $(T_{\alpha+1})_v \in \mathcal{D}_\alpha$. For limit $\gamma \in \kappa$ we already constructed the descending chain of conditions $\langle T_\xi \mid \xi \in \gamma\rangle$, so we let $T_\gamma = \bigcap_{\xi<\gamma} T_\xi$.

Let T_κ be the fusion limit of $\langle T_\alpha \mid \alpha \in \kappa\rangle$, then by construction we see that if $v \in \mathrm{Split}_{\alpha+1}(T_\kappa)$, then $(T_\kappa)_v$ decides $\dot\tau \quad \alpha$.

Finally, note that $\{s_\alpha(T_\kappa) \mid \alpha \in \kappa\}$ is club, so $C = \{\alpha \in \kappa \mid \alpha = s_\alpha(T_\kappa)\}$ is club as well. We define T' to be a collapse of T_κ on $\kappa \setminus C$. Let $u \in T'$ with $\mathrm{dom}(u) = \alpha \in C$, then for any $\beta < \alpha$, we see that $s_\beta(T_\kappa) < s_\alpha(T_\kappa) = \alpha$, and therefore $(T_\kappa)_{u \restriction (s_\beta(T_\kappa)+1)}$ decides $\dot\tau \restriction \beta$. But this implies that $(T_\kappa)_u$ decides $\dot\tau \restriction \alpha$, and hence also $(T')_u$ decides $\dot\tau \restriction \alpha$. On the other hand, if $u \in T'$ with $\mathrm{dom}(u) = \beta \notin C$, let $\alpha \in C$ be minimal with $\beta < \alpha$, then $s_\beta(T_\kappa) < \alpha$. Let $v \in T'$ be such that $u \subseteq v$ and $\mathrm{dom}(v) = s_\beta(T_\kappa) + 1$, then $(T_\kappa)_v$ decides $\dot\tau \restriction \beta$. But, since $\bigcup_{\beta \leq \xi < \alpha} \mathrm{Lev}_\xi(T')$ contains no splitting nodes in T' (because T' is a collapse of T_κ on $\kappa \setminus C$), we see that $(T')_u = (T')_v$, hence $(T')_u$ decides $\dot\tau \restriction \beta$. □

Corollary 6.1.11

$\mathsf{Q}^{b,h}_\kappa$ preserves κ^+. ◁

Proof. Let $\dot\tau$ be a name and $T \in \mathsf{Q}^{b,h}_\kappa$ be such that $T \Vdash$ "$\dot\tau : \kappa \to \kappa^+$" and T reads $\dot\tau$ early. For each $\alpha \in \kappa$ there is a set $B_\alpha \subseteq \kappa^+$ with $|B_\alpha| \leq |\mathrm{Lev}_\alpha(T)| < \kappa$ such that $T \Vdash$ "$\dot\tau(\alpha) \in B_\alpha$", thus $T \Vdash$ "$\dot\tau[\kappa] \subseteq \bigcup_{\alpha\in\kappa} B_\alpha$". Therefore $T \Vdash$ "$\dot\tau$ is not surjective". □

We will now look at the effect that $\mathsf{Q}^{b,h}_\kappa$ has on the cardinality of antiavoidance numbers. We first note that the generic slalom added by $\mathsf{Q}^{b,h}_\kappa$ does not antilocalise any $f \in \prod b$ from the ground model, hence that it is a (b,h)-antilocalising κ-real. By adding many such generics we can increase $\mathfrak{d}^{b,h}_\kappa(\ni^\infty)$.

Lemma 6.1.12 — *cf. [KM22, Lemma 3.3] for ${}^\omega\omega$*

Let $\varphi_G \in \mathrm{Loc}^{b,h}_\kappa$ be $\mathsf{Q}^{b,h}_\kappa$-generic over $\mathbf{V}$, let $h' \leq^* b' \in {}^\kappa\kappa$ be cofinal increasing cardinal functions and let $S \subseteq \kappa$ be stationary such that $h(\alpha) \leq h'(\alpha) \leq b'(\alpha) \leq b(\alpha)$ for all $\alpha \in S$. If $\varphi'_G \in \mathrm{Loc}^{b'}_{h'}$ satisfies $\varphi'_G(\alpha) = \varphi_G(\alpha) \cap b'(\alpha)$ for all $\alpha \in S$, then φ'_G is (b',h')-antilocalising over $\mathbf{V}$. ◁

Note that this holds specifically for $b = b'$ and $h = h'$, in which case $\varphi'_G = \varphi_G$.

Proof. Working in $\mathbf{V}$, fix some $f \in \prod b'$ and $T \in (\mathsf{Q}^{b,h}_\kappa)^*$ and $\alpha_0 \in \kappa$. Since $C = \{s_\xi(T) \mid \xi \in \kappa\}$ is a club set, we can choose $\alpha > \alpha_0$ such that $\alpha \in S \cap C$. We see that $\|\mathrm{suc}(u,T)\|_{b,\alpha} > 1$ for any $u \in \mathrm{Lev}_\alpha(t)$, since u is splitting. Definition 6.1.1 implies that there is some $v \in \mathrm{suc}(u,T)$ such that $\{f(\alpha)\} \subseteq v(\alpha)$. Then clearly $(T)_v \Vdash$ "$f(\alpha) \in \dot\varphi_G(\alpha)$", and because $f(\alpha) \in b'(\alpha)$ it follows that $(T)_v \Vdash$ "$f(\alpha) \in \dot\varphi'_G(\alpha)$". Since α_0 was arbitrary, $\mathbf{V}[G] \vDash$ "$f \in^\infty \varphi'_G$". □

On the other hand, we can give assumptions on the parameters b', h' such that the cardinal $\mathfrak{d}^{b',h'}_\kappa(\ni^\infty)$ is preserved by $\mathsf{Q}^{b,h}_\kappa$. We first give assumptions such that a forcing notion $\mathsf{Q}^{b,h}_\kappa$ has the (b',h')-Laver property.

Lemma 6.1.13 — *cf. [KM22, Lemma 3.13] for ${}^\omega\omega$*

Let $b, h, b', h' \in {}^\kappa\kappa$ be increasing cofinal cardinal functions such that for almost all $\alpha \in \kappa$ we have $\left|\prod_{\xi\leq\alpha}[b(\xi)]^{<h(\xi)}\right| < h'(\alpha) \leq b'(\alpha)$, then $\mathsf{Q}^{b,h}_\kappa$ has the (b',h')-Laver property. ◁

Proof. Let $T \in \mathsf{Q}^{b,h}_\kappa$ be such that $T \Vdash$ "$\dot f \in \prod b'$", then we want to find $\varphi \in \mathrm{Loc}^{b',h'}_\kappa$ and $T' \leq T$ such that $T' \Vdash$ "$\dot f \in^* \check\varphi$". Using Lemma 6.1.10, let $T' \in (\mathsf{Q}^{b,h}_\kappa)^*$ be such that $T' \leq T$ and T' reads $\dot f$ early, then we can define a unique $y_v \in b'(\alpha)$ for each $v \in \mathrm{Lev}_{\alpha+1}(T')$ with $(T')_v \Vdash$ "$\dot f(\alpha) = y_v$". Note that $|\mathrm{Lev}_{\alpha+1}(T')| \leq \left|\prod_{\xi\leq\alpha}[b(\xi)]^{<h(\xi)}\right| < h'(\alpha)$. Therefore if $\varphi : \alpha \mapsto \{y_v \mid v \in \mathrm{Lev}_{\alpha+1}(T')\}$, then $\varphi \in \mathrm{Loc}^{b',h'}_\kappa$ and by construction we see that $T' \Vdash$ "$\dot f \in^* \check\varphi$". □

Corollary 6.1.14

Under the assumptions of the lemma, $\mathsf{Q}_\kappa^{b,h}$ does not add a (b',h')-avoiding κ-real. $\triangleleft$

The (b,h)-Laver property is related to the preservation of $\mathfrak{d}_\kappa^{b,h}(\in^*)$, as we saw in Lemma 4.2.11 and Scenario 2 of Remark 4.2.2, since not adding a (b,h)-avoiding κ-real will imply that $\mathfrak{d}_\kappa^{b,h}(\in^*) \leq (2^\kappa)^{\mathbf{V}}$. We are, however, interested in antiavoidance, and wish to find a property of $\mathsf{Q}_\kappa^{b,h}$ such that $\mathfrak{d}_\kappa^{\bar{b},\bar{h}}(\ni^\infty)$ is preserved for some $\bar{b},\bar{h}$. The following lemma gives us the property we need, using the Tukey connection from Theorem 3.3.9.

Lemma 6.1.15 — *cf. [KM22, Lemma 3.15] for $^\omega\omega$*

Let $b,h,b',h',\tilde{b},\tilde{h} \in {}^\kappa\kappa$ be increasing cofinal cardinal functions such that for almost all $\alpha \in \kappa$

$$\left|\prod_{\xi\leq\alpha}[b(\xi)]^{<h(\xi)}\right| < h'(\alpha) \leq \tilde{b}(\alpha)^{<\tilde{h}(\alpha)} \leq b'(\alpha) \quad \text{and}$$
$$\tilde{h}(\alpha)\cdot h'(\alpha) < \tilde{b}(\alpha).$$

Then $\mathsf{Q}_\kappa^{b,h}$ does not add a $(\bar{b},\bar{h})$-antilocalising κ-real. That is, if $\dot{\psi}$ is a $\mathsf{Q}_\kappa^{b,h}$-name and $T \in \mathsf{Q}_\kappa^{b,h}$ is such that T "$\dot{\psi} \in \mathrm{Loc}_\kappa^{\tilde{b},\tilde{h}}$", then there is $g \in \prod \tilde{b}$ and $T' \leq T$ such that T' "$g \in^\infty \dot{\psi}$". $\triangleleft$

Proof. Working in the extension, we define the functions ρ_-,ρ_+ witnessing that $\mathbf{AL}_{\tilde{b},\tilde{h}} \preceq \mathbf{L}_{b',h'}$ and the injections ι_α for $\alpha \in \kappa$ as in the proof of Theorem 3.3.9. Note that if we assume that $\iota_\alpha \in \mathbf{V}$ for each α, then due to the constructive definition of ρ_- and ρ_+, it follows that ρ_- $(\mathrm{Loc}_\kappa^{\tilde{b},\tilde{h}})^{\mathbf{V}}$ and ρ_+ $(\mathrm{Loc}_\kappa^{b',h'})^{\mathbf{V}}$ are in the ground model.

Let $\dot{f}$ be a name for $\rho_-(\dot{\psi})$ and by Lemma 6.1.13, let $\varphi \in \mathrm{Loc}_\kappa^{b',h'}$ and $T' \leq T$ be such that T' "$\dot{f} \in^* \varphi$". Let $g = \varphi_+(\varphi) \in \prod \tilde{b}$, then Theorem 3.3.9 shows that T' "$g \in^\infty \dot{\psi}$". $\square$

6.2. Products of $\mathsf{Q}_\kappa^{b,h}$

If we wish to increase $\mathfrak{d}_\kappa^{b,h}(\ni^\infty)$, we will have to add many new (b,h)-antilocalising κ-reals. We will do so using a $\leq\kappa$-support product of forcing notions of the form $\mathsf{Q}_\kappa^{b,h}$. We could use a $\leq\kappa$-support iteration as well, but this has the drawback that we cannot increase 2^κ past κ^{++}. We will show that the product behaves nicely, especially that Lemma 6.1.15 is preserved under products, and that adding many $\mathsf{Q}_\kappa^{b,h}$-generic elements will indeed increase the size of $\mathfrak{d}_\kappa^{b,h}(\ni^\infty)$.

For the remainder of this section, we will fix some set of ordinals $\mathcal{A}$ and $b_\zeta,h_\zeta \in {}^\kappa\kappa$ for each $\zeta \in \mathcal{A}$ and we fix the abbreviations $\mathsf{Q}_\zeta = \mathsf{Q}_\kappa^{b_\zeta,h_\zeta}$ and $\mathsf{Q}_\zeta^* = (\mathsf{Q}_\kappa^{b_\zeta,h_\zeta})^*$. We define the $\leq\kappa$-support products $\overline{\mathsf{Q}} = \prod_{\zeta\in\mathcal{A}}^{\leq\kappa}\mathsf{Q}_\zeta$ and $\overline{\mathsf{Q}}^* = \prod_{\zeta\in\mathcal{A}}^{\leq\kappa}\mathsf{Q}_\zeta^*$. Since each Q_ζ^* densely embeds in Q_ζ, it is easy to see that $\overline{\mathsf{Q}}^*$ densely embeds in $\overline{\mathsf{Q}}$. We will often implicitly assume without mention that all conditions are in $\overline{\mathsf{Q}}^*$.

Lemma 6.2.1

If $\kappa^+ = 2^\kappa$, then $\overline{\mathsf{Q}}$ is $<\kappa$-closed and $<\kappa^{++}$-c.c. $\triangleleft$

Proof. By Theorems 4.1.15 and 4.1.17. $\square$

Lemma 6.2.2
$\overline{\mathsf{Q}}^*$ is closed under generalised fusion, i.e. for any generalised fusion sequence $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa \rangle$ there exists $p \leq_{\alpha, Z_\alpha} p_\alpha$ with $\operatorname{supp}(p) = \bigcup_{\alpha\in\kappa} \operatorname{supp}(p_\alpha)$. $\lhd$

Proof. Let $S = \bigcup_{\alpha\in\kappa} \operatorname{supp}(p_\alpha)$, then $\bigcup_{\alpha\in\kappa} Z_\alpha = S$, so for each $\zeta \in S$ we can fix $\alpha_\zeta \in \kappa$ such that $\zeta \in Z_{\alpha_\zeta}$. Then also $\zeta \in Z_\alpha$ for any $\alpha \geq \alpha_\zeta$, since $Z_\alpha \supseteq Z_{\alpha_\zeta}$. If $\alpha_\zeta \leq \alpha \leq \beta < \kappa$, then $p_\beta \leq_{\alpha, Z_\alpha} p_\alpha$, and thus by $\zeta \in Z_\alpha$ we see that $p_\beta(\zeta) \leq_\alpha p_\alpha(\zeta)$. Therefore $\langle p_{\alpha_\zeta+\alpha}(\zeta) \mid \alpha \in \kappa \rangle$ is a fusion sequence in Q^*_ζ and we can define $p(\zeta) = \bigcap_{\alpha\in\kappa} p_{\alpha_\zeta+\alpha}(\zeta)$, then $p(\zeta) \in \mathsf{Q}^*_\zeta$ by Lemma 6.1.6. $\square$

For $p \in \overline{\mathsf{Q}}$ and $\alpha \in \kappa$, we define the set of ***possibilities*** $\operatorname{poss}(p, {<}\alpha)$ to be the set of functions η with domain $\operatorname{supp}(p)$ such that $\eta(\zeta) \in \operatorname{Lev}_\alpha(p(\zeta))$ for all $\zeta \in \operatorname{supp}(p)$. We similarly define $\operatorname{poss}(p, {\leq}\alpha) = \operatorname{poss}(p, {<}\alpha + 1)$. Remember that $(p(\zeta))_u$ is the subtree of $p(\zeta)$ generated by u. If $\eta \in \operatorname{poss}(p, {<}\alpha)$, we define $\eta \wedge p$ to be the condition with $(\eta \wedge p)(\zeta) = (p(\zeta))_{\eta(\zeta)}$ for $\zeta \in \operatorname{supp}(p)$ and $(\eta \wedge p)(\zeta) = \mathbb{1}_\zeta$ otherwise. We sometimes abuse this notation also to define $\eta \wedge q$ for $q \leq p$ with larger support, where we let $(\eta \wedge q)(\zeta) = q(\zeta)$ for all $\zeta \in \operatorname{supp}(q) \setminus \operatorname{supp}(p)$.

For $p \in \overline{\mathsf{Q}}^*$, we define $\operatorname{Split}(p) = \bigcup_{\zeta\in\operatorname{supp}(p)} \{s_\alpha(p(\zeta)) \mid \alpha \in \kappa\}$ and let $\langle \overline{s}_\alpha(p) \mid \alpha \in \kappa \rangle$ be the strictly increasing enumeration of $\operatorname{Split}(p)$. Let $\mathcal{Z}^p(\alpha) = \{\zeta \in \operatorname{supp}(p) \mid \exists \xi (s_\xi(p(\zeta)) = \alpha)\}$.

We call $p \in \overline{\mathsf{Q}}^*$ ***modest*** if for any $\alpha \in \kappa$ we have $|\operatorname{poss}(p, {<}\alpha)| < \kappa$ and $|\mathcal{Z}^p(\alpha)| \leq \alpha$, and moreover $|\mathcal{Z}^p(\overline{s}_\alpha(p))| = 1$ in case α is successor.

Lemma 6.2.3 — *cf. [KM22, Lemma 5.2] for* $^\omega\omega$
The set of modest conditions is dense in $\overline{\mathsf{Q}}^*$ (hence in $\overline{\mathsf{Q}}$ as well). $\lhd$

Proof. Let $p \in \overline{\mathsf{Q}}^*$. We will assume for convenience (and without loss of generality) that $|\operatorname{supp}(p)| = \kappa$. Enumerate $\operatorname{supp}(p)$ as $\langle \zeta_\alpha \mid \alpha \in \kappa \rangle$ and let $v_\alpha \in \operatorname{Lev}_{\alpha+1}(p(\zeta_\alpha))$ be arbitrary. We define q as

$$\begin{aligned} q(\zeta_\alpha) &= (p(\zeta_\alpha))_{v_\alpha} && \text{for all } \alpha \in \kappa, \\ q(\zeta) &= \mathbb{1}_\zeta && \text{if } \zeta \notin \operatorname{supp}(p). \end{aligned}$$

Then $q \leq p$ and $|\operatorname{poss}(q, {<}\alpha)| < \kappa$ and $|\mathcal{Z}^q(\alpha)| \leq \alpha$ for all $\alpha \in \kappa$. Note that if $r \leq q$ is such that $\operatorname{supp}(r) = \operatorname{supp}(q)$, then $|\operatorname{poss}(r, {<}\alpha)| < \kappa$ and $|\mathcal{Z}^r(\alpha)| \leq \alpha$ still hold for all $\alpha \in \kappa$.

We will define $r \leq q$ such that r is modest. Note that $C = \{\overline{s}_\alpha(q) \mid \alpha \in \kappa \text{ is limit}\}$ contains a club set. Let $A_\zeta = \{\overline{s}_\alpha(q) \mid \alpha \in \kappa \text{ is successor} \wedge \zeta \neq \min(\mathcal{Z}^q(\overline{s}_\alpha(q)))\}$. For any $\zeta \in \operatorname{supp}(q)$, we define $r(\zeta)$ to be a collapse of $q(\zeta)$ on A_ζ, and for any $\zeta \notin \operatorname{supp}(q)$ we let $r(\zeta) = \mathbb{1}_\zeta$. It is clear from this construction that $\operatorname{Split}(r) = \operatorname{Split}(q)$, and that $\zeta \in \mathcal{Z}^r(\overline{s}_\alpha(r))$ implies that α is limit or that $\zeta = \min(\mathcal{Z}^q(\overline{s}_\alpha(q)))$. In other words, r is modest.

Finally $r(\zeta) \in \mathsf{Q}^*_\zeta$ is implied by $p(\zeta) \in \mathsf{Q}^*_\zeta$, because for each $\alpha \in \kappa$ there is $\beta \geq \alpha$ such that $s_\alpha(r(\zeta)) = s_\beta(p(\zeta))$, and for any $\alpha \in \operatorname{Split}(r)$ and $\zeta \in \mathcal{Z}^r(\alpha)$ we have $\operatorname{suc}(u, r(\zeta)) = \operatorname{suc}(u, p(\zeta))$ for all $u \in \operatorname{Lev}_\alpha(r)$. Notably, for the norm we see that

$$\|\operatorname{suc}(u, r(\zeta))\|_{b,\operatorname{dom}(u)} = \|\operatorname{suc}(u, p(\zeta))\|_{b,\operatorname{dom}(u)} \geq \beta \geq \alpha. \qquad \square$$

The reason we are interested in modest conditions, is that modest conditions behave similarly to conditions of the single forcing notion $\mathsf{Q}^{b,h}_{\kappa}$. By using modest conditions, the number of possibilities up to a certain height α is bound below κ, which will be crucial in preservation of the Laver property (Lemma 6.2.8).

We will first use modest conditions to generalise Lemma 6.1.7 to products. In order to state the lemma, we define one more ordering on $\overline{\mathsf{Q}}^*$ as follows: $q \leq^*_\alpha p$ if

- $q \leq p$ and
- $\mathrm{Lev}_{\overline{s}_\alpha(p)}(q(\zeta)) = \mathrm{Lev}_{\overline{s}_\alpha(p)}(p(\zeta))$ for all $\zeta \in \mathrm{supp}(p)$ and
- $s_0(q(\zeta)) > \overline{s}_\alpha(p)$ for any $\zeta \in \mathrm{supp}(q) \setminus \mathrm{supp}(p)$.

Lemma 6.2.4 — *cf. [KM22, Lemma 5.4] for* $^\omega\omega$

If $\alpha \in \kappa$, $p \in \overline{\mathsf{Q}}^*$ is modest and $\mathcal{D} \subseteq \overline{\mathsf{Q}}$ is open dense, then there exists $q \in \overline{\mathsf{Q}}^*$ with $q \leq^*_\alpha p$ such that for any $\eta \in \mathrm{poss}(q, \leq\overline{s}_\alpha(q))$ we have $\eta \wedge q \in \mathcal{D}$. ◁

Proof. By assumption p is modest, thus if $\langle \eta_\xi \mid \xi \in \mu\rangle$ enumerates $\mathrm{poss}(p, \leq\overline{s}_\alpha(p))$, then $\mu < \kappa$. We create a descending sequence of conditions $\langle p_\xi \mid \xi \in \mu\rangle$. Let $p_0 = p$. If $\gamma < \mu$ is limit, let $p_\gamma = \bigwedge_{\xi<\gamma} p_\xi$ using Lemma 6.2.1. If p_ξ has been defined, let $p'_{\xi+1} \leq \eta_\xi \wedge p_\xi$ be such that $p'_{\xi+1} \in \mathcal{D}$. We define $p_{\xi+1}$ from $p'_{\xi+1}$ in a pointwise manner. If $\zeta \in \mathrm{supp}(p)$, we let

$$p_{\xi+1}(\zeta) = p'_{\xi+1}(\zeta) \cup \bigcup \left\{ (p_\xi(\zeta))_v \mid v \in \mathrm{Lev}_{\overline{s}_\alpha(p)+1}(p_\xi(\zeta)) \text{ and } v \neq \eta_\xi(\zeta) \right\}.$$

In plain words, we keep $p_{\xi+1}(\zeta)$ almost equal to $p_\xi(\zeta)$, except that we replace the part of $p_\xi(\zeta)$ that extends $\eta_\xi(\zeta)$ with the tree $p'_{\xi+1}(\zeta)$. On the other hand, if $\zeta \in \mathrm{supp}(p'_{\xi+1}) \setminus \mathrm{supp}(p)$, we let $p_{\xi+1}(\zeta) = (p'_{\xi+1}(\zeta))_u$ for some $u \in \mathrm{Lev}_{\overline{s}_\alpha(p)+1}(p'_{\xi+1}(\zeta))$. Finally if $\zeta \notin \mathrm{supp}(p'_{\xi+1})$ we let $p_{\xi+1}(\zeta) = \mathbb{1}_\zeta$.

Note that $p'_{\xi+1}(\zeta) = (p'_{\xi+1}(\zeta))_{\eta_\xi(\zeta)} = (p_{\xi+1}(\zeta))_{\eta_\xi(\zeta)}$ for all $\zeta \in \mathrm{supp}(p)$, and thus $\eta_\xi \wedge p_{\xi+1} = p'_{\xi+1}$.

Finally define $q = \bigwedge_{\xi\in\mu} p_\xi$, then we see that $q \leq^*_\alpha p$ and $\eta \wedge q \leq \eta \wedge p_{\xi+1} = p'_{\xi+1} \in \mathcal{D}$ for each $\eta \in \mathrm{poss}(q, \leq\overline{s}_\alpha(q)) = \mathrm{poss}(p, \leq\overline{s}_\alpha(p))$. □

We can also generalise the notion of early reading in the most apparent sense.

Definition 6.2.5

Let $p \in \overline{\mathsf{Q}}$ and $\dot{\tau}$ be a $\overline{\mathsf{Q}}$-name such that p “$\dot{\tau} : \kappa \to \mathbf{V}$”, then we say that p reads $\dot{\tau}$ *early* iff $\eta \wedge p$ decides $\dot{\tau}$ α for every $\alpha \in \kappa$ and $\eta \in \mathrm{poss}(p, <\alpha)$. ◁

Lemma 6.2.6 — *cf. [KM22, Lemma 5.6] for* $^\omega\omega$

Let $p \in \overline{\mathsf{Q}}$ and $\dot{\tau}$ a $\overline{\mathsf{Q}}$-name such that p “$\dot{\tau} : \kappa \to \mathbf{V}$”. Then there exists $q \leq p$ with $q \in \overline{\mathsf{Q}}^*$ such that q reads $\dot{\tau}$ early. ◁

Proof. Let $\mathcal{D}_\alpha = \left\{ q \in \overline{\mathsf{Q}} \mid q \text{ decides } \dot{\tau} \quad \alpha \right\}$, and note that $\mathcal{D}_\alpha$ is open dense for each $\alpha \in \kappa$. We prove the lemma by constructing $q' \leq p$ such that $\eta \wedge q' \in \mathcal{D}_\alpha$ for all $\eta \in \mathrm{poss}(q', \leq\alpha)$. We claim that this is sufficient: define q such that $q(\zeta)$ is a collapse of $q'(\zeta)$ on $\{\overline{s}_\alpha(q') \mid \alpha \text{ is successor}\}$ for each $\zeta \in \mathrm{supp}(q')$ and $q(\zeta) = \mathbb{1}_\zeta$ otherwise, then q reads $\dot{\tau}$ early.

The condition q' will be the limit of a generalised fusion sequence $\langle (p_\alpha, Z_\alpha) \mid \alpha \in \kappa\rangle$. Each p_α will be modest and have the following property:

$$(\star_\alpha) \qquad s_\alpha(p_\alpha(\zeta)) < s_0(p_\alpha(\zeta')) \text{ for all } \zeta \in Z_\alpha \text{ and } \zeta' \in \operatorname{supp}(p_\alpha) \setminus Z_\alpha.$$

Given p_α satisfying $(\star_\alpha)$, let $\beta_\alpha = \sup\{s_\alpha(p_\alpha(\zeta)) \mid \zeta \in Z_\alpha\}$ and suppose $p_{\alpha+1} \leq^*_{\beta_\alpha} p_\alpha$, then it follows from $(\star_\alpha)$ that $p_{\alpha+1} \leq_{\alpha, Z_\alpha} p_\alpha$.

Firstly, we let $p_0 \leq p$ be modest such that $p_0 \in \mathcal{D}_{\bar{s}_0(p_0)}$. This can be easily achieved by letting $p_0^0 \leq p$ be modest, finding modest $p_0^{n+1} \leq p_0^n$ such that $p_0^{n+1} \in \mathcal{D}_{\bar{s}_0(p_0^n)}$ and letting $p_0 = \bigwedge_{n\in\omega} p_0^n$. We set $Z_0 = \mathcal{Z}^{p_0}(\bar{s}_0(p_0))$, then $|Z_0| < \kappa$ by modesty and p_0 satisfies $(\star_0)$. We may also assume that $|Z_0|$ is infinite and that $|\operatorname{supp}(p_0)| = \kappa$.

Next, for limit γ, we have $Z_\gamma = \bigcup_{\alpha\in\gamma} Z_\alpha$ and $\hat{p}_\gamma = \bigwedge_{\alpha<\gamma} p_\gamma$. We let $p_\gamma \leq_{\gamma, Z_\gamma} \hat{p}_\gamma$ be such that it has $(\star_\gamma)$. This is possible, since we may keep $p_\gamma(\zeta) = \hat{p}_\gamma(\zeta)$ for all $\zeta \in Z_\gamma$ and thus trivially have $p_\gamma \leq_{\gamma, Z_\gamma} \hat{p}_\gamma$. The construction of the successor step will show that $\beta_\gamma = s_\gamma(p_\gamma(\zeta)) = s_\gamma(p_\gamma(\zeta'))$ for any $\zeta, \zeta' \in Z_\gamma$. If $\eta \in \operatorname{poss}(p_\gamma, {<}\beta_\gamma)$ and $\delta < \beta_\gamma$, then there is $\alpha < \gamma$ such that $\eta \wedge p_\alpha \in \mathcal{D}_\delta$, therefore $\eta \wedge p_\gamma \in \mathcal{D}_{\beta_\gamma}$.

Finally we construct $p_{\alpha+1}$ from p_α. Let $\lambda = |Z_\alpha|$ and enumerate Z_α as $\langle \zeta_\xi \mid \xi < \lambda\rangle$. We use bookkeeping to fulfil the promise that $\bigcup_{\alpha\in\kappa} Z_\alpha = \bigcup_{\alpha\in\kappa} \operatorname{supp}(p_\alpha)$, thereby setting $Z_{\alpha+1} = Z_\alpha \cup \{\zeta_\lambda\}$ for some appropriate $\zeta_\lambda \in \operatorname{supp}(p_\alpha) \setminus Z_\alpha$. We construct a descending sequence of conditions $\langle p_\alpha^\xi \mid \xi \leq \lambda + \alpha + 1\rangle$ by recursion over a strictly increasing sequence of ordinals $\langle \delta_\alpha^\xi \mid \xi \leq \lambda + \alpha + 1\rangle$, to obtain the following properties (which we will clarify below):

(i) $p_\alpha^0 \leq^*_{\beta_\alpha} p_\alpha$, where $\beta_\alpha = \sup\{s_\alpha(p_\alpha(\zeta)) \mid \zeta \in Z_\alpha\}$,
(ii) $p_\alpha^{\xi'} \leq^*_{\delta_\alpha^\xi} p_\alpha^\xi$ for all $\xi < \xi' \leq \lambda + \alpha + 1$,
(iii) $\delta_\alpha^\xi = s_{\alpha+1}(p_\alpha^\xi(\zeta_\xi))$ for all $\xi < \lambda$,
(iv) $\delta_\alpha^\xi < s_{\alpha+1}(p_\alpha^\xi(\zeta_{\xi'}))$ for all $\xi < \xi' < \lambda$,
(v) $\delta_\alpha^\xi < s_0(p_\alpha^\xi(\zeta))$ for all $\xi < \lambda$ and $\zeta \in \operatorname{supp}(p_\alpha^\xi) \setminus Z_\alpha$,
(vi) $\delta_\alpha^{\lambda+\epsilon} = s_\epsilon(p_\alpha^{\lambda+\epsilon}(\zeta_\lambda))$ for all $\epsilon < \alpha$,
(vii) $\delta_\alpha^{\lambda+\epsilon} < s_{\alpha+2}(p_\alpha^{\lambda+\epsilon}(\zeta_\xi))$ for all $\epsilon < \alpha$ and $\xi < \lambda$,
(viii) $\delta_\alpha^{\lambda+\epsilon} < s_0(p_\alpha^{\lambda+\epsilon}(\zeta))$ for all $\epsilon < \alpha$ and $\zeta \in \operatorname{supp}(p_\alpha^{\lambda+\epsilon}) \setminus Z_{\alpha+1}$,
(ix) For all $\xi \leq \lambda + \alpha + 1$ and any $\eta \in \operatorname{poss}(p_\alpha^\xi, {\leq}\delta_\alpha^\xi)$ we have $\eta \wedge p_\alpha^\xi \in \mathcal{D}_{\delta_\alpha^\xi}$.

We will set $p_{\alpha+1} = p_\alpha^{\lambda+\alpha+1}$. By construction $p_{\alpha+1}$ satisfies $(\star_{\alpha+1})$ and $p_{\alpha+1} \leq_{\alpha, Z_\alpha} p_\alpha$.

The result of this construction is summarised in Figure 6.1. Let us clarify this diagram and the recursive construction. The initial splitting levels $s_\alpha(p_\alpha(\zeta_\xi))$ with $\xi < \lambda$ occur below β_α and are left unmodified during the entire construction. The ordinals δ_α^ξ give us the height of $s_{\alpha+1}(p_{\alpha+1}(\zeta_\xi))$ with $\xi < \lambda$, and the ordinals $\delta_\alpha^{\lambda+\epsilon}$ give the height of $s_\epsilon(p_{\alpha+1}(\zeta_\lambda))$. For any other $\zeta \in \operatorname{supp}(p_{\alpha+1}) \setminus Z_{\alpha+1}$ the splitting starts strictly above $\beta_{\alpha+1}$, that is, $s_0(p_{\alpha+1}(\zeta)) > s_{\alpha+1}(p_{\alpha+1}(\zeta_\lambda))$. At step ξ of the recursive construction, we decide on δ_α^ξ and hence on the splitting levels up to δ_α^ξ, making sure that the splitting levels we have not considered yet occur at a strictly higher level. We use Lemma 6.2.4 to make sure we satisfy (ix) without disturbing the splitting levels up to δ_α^ξ. This automatically gives us modesty as well. □

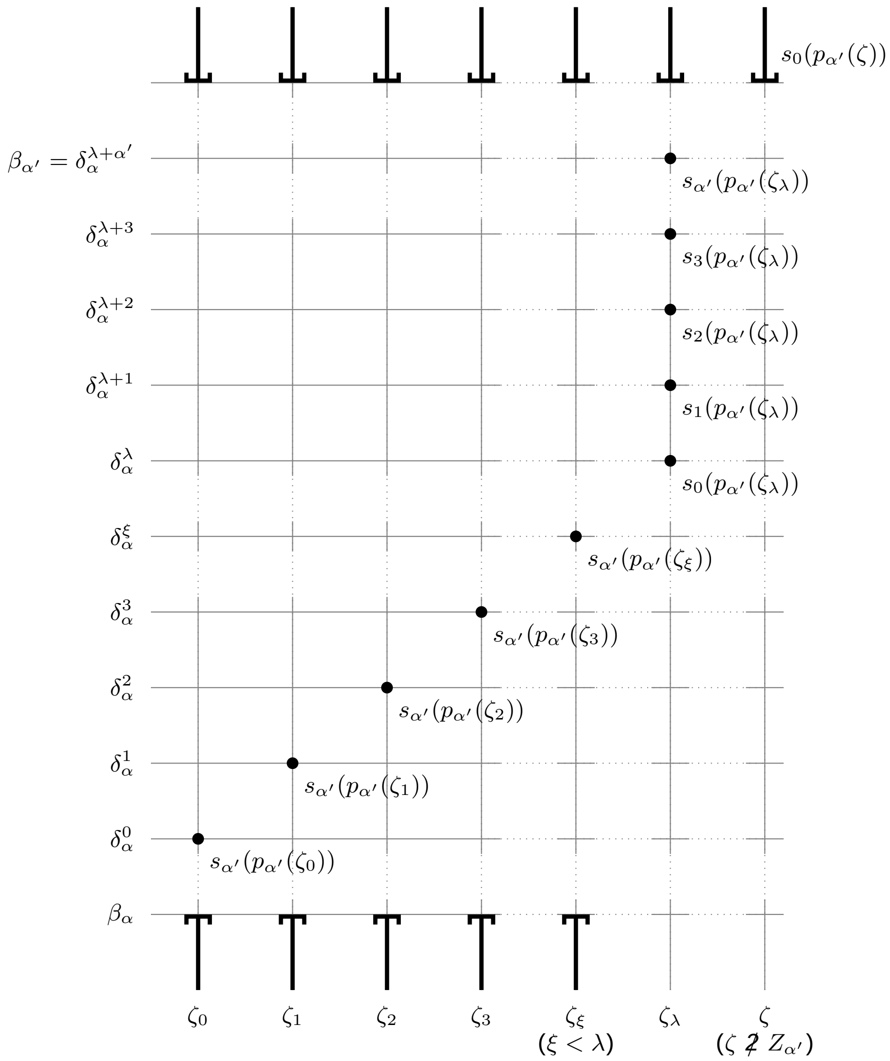


Figure 6.1: The structure of Split$(p_{\alpha+1})$ in the proof of Lemma 6.2.6. The thicker lines and dots show the occurrences of splitting nodes for each index in the support (horizontal axis) and levels of the condition (vertical axis). We write $\alpha + 1$ as α' for brevity.

We are now ready to give the last two lemmas necessary to prove our consistency result. Lemma 6.2.7 is a generalisation of Lemma 6.1.12 and shows that we can increase $\mathsf{d}_\kappa^{b,h}(\ni^\infty)$ with $\overline{\mathsf{Q}}$, and Lemma 6.2.8 gives us the preservation of the Laver property, which is a generalisation of Lemma 6.1.13. One could also compare these two lemmas to Lemmas 5.2.3 and 5.2.5.

Lemma 6.2.7

Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of ordinals, with $\mathcal{B}^c = \mathcal{A} \setminus \mathcal{B}$, and let $\langle h_\zeta, b_\zeta \mid \zeta \in \mathcal{A}\rangle$ be a sequence of cofinal increasing cardinal functions such that $h_\zeta \leq^* b_\zeta$ for all $\zeta \in \mathcal{A}$. Let $\overline{\mathsf{Q}} = \prod_{\zeta\in\mathcal{A}}^{\leq\kappa} \mathsf{Q}_\zeta$, let G be $\overline{\mathsf{Q}}$-generic over $\mathbf{V}$ and $\mathbf{V}$ “ $2^\kappa = \kappa^+$ ”. If $h' \leq^* b' \in {}^\kappa\kappa$ are cofinal increasing cardinal functions such that for each $\zeta \in \mathcal{B}$ there exists a stationary set $S_\zeta \subseteq \kappa$ such that $h_\zeta(\alpha) \leq h'(\alpha) \leq b'(\alpha) \leq b_\zeta(\alpha)$ for all $\alpha \in S_\zeta$. Then $\mathbf{V}[G]$ “ $|\mathcal{B}| \leq \mathsf{d}_\kappa^{b',h'}(\ni^\infty)$ ”. ◁

Proof. The lemma is trivial if $|\mathcal{B}| \leq \kappa^+$, thus we assume $\kappa^{++} \leq |\mathcal{B}|$.

We work in $\mathbf{V}[G]$. Let $\mu < |\mathcal{B}|$ and let $\{f_\xi \mid \xi < \mu\} \subseteq \prod b'$, then we want to describe some $\varphi \in \mathrm{Loc}_\kappa^{b',h'}$ such that $f_\xi \in^\infty \varphi$ for each $\xi < \mu$. Since $\overline{\mathbb{Q}}$ is $<\kappa^{++}$-c.c., we could find $A_\xi \subseteq \mathcal{A}$ with $|A_\xi| \leq \kappa^+$ for each $\xi < \mu$ such that $f_\xi \in \mathbf{V}[G \restriction A_\xi]$. Since $|\mathcal{B}| > \mu \cdot \kappa^+$, we may fix some $\beta \in \mathcal{B} \setminus \bigcup_{\xi<\mu} A_\xi$ for the remainder of this proof. Let $\varphi_\beta = \bigcap_{p \in G} p(\beta)$ be the $\mathbb{Q}_\beta$-generic κ-real added by the β-th term of the product $\overline{\mathbb{Q}}$, and let $\varphi' \in \mathrm{Loc}_\kappa^{b',h'}$ be such that $\varphi'(\alpha) = \varphi_\beta(\alpha) \cap b'(\alpha)$ for each $\alpha \in S_\beta$.

Continuing the proof in the ground model, let $\dot\varphi'$ be a $\overline{\mathbb{Q}}$-name for φ' and $\dot f_\xi$ be a $\overline{\mathbb{Q}}$-name for f_ξ, let $p \in \overline{\mathbb{Q}}^*$ and $\alpha_0 < \kappa$. We want to find some $\alpha \geq \alpha_0$ and $q \leq p$ such that $q \Vdash$ "$\dot f_\xi(\alpha) \in \dot\varphi'(\alpha)$".

We now reason as in Lemma 6.1.12. Let $C = \{s_\xi(p(\beta)) \mid \xi \in \kappa\}$, then C is club. Therefore, there exists $\alpha \in S_\beta \cap C$ with $\alpha \geq \alpha_0$. Choose some $p_0 \leq p$ such that $p_0(\beta) = p(\beta)$ and such that there is a $\gamma \in b'(\alpha)$ for which $p_0 \Vdash$ "$\dot f_\xi(\alpha) = \gamma$". This is possible, since $f_\xi \in \mathbf{V}[G \restriction A_\xi]$ and $\beta \notin A_\xi$, therefore we could find $p_0' \in \overline{\mathbb{Q}} \restriction A_\xi$ with $p_0' \leq p \restriction A_\xi$ and γ with the aforementioned property, and then let $p_0(\eta) = p_0'(\eta)$ if $\eta \in A_\xi$ and $p_0(\eta) = p(\eta)$ otherwise.

Note that $\alpha \in C$ implies $\|\mathrm{suc}(u, p_0(\beta))\|_{b_\beta,\alpha} = \|\mathrm{suc}(u, p(\beta))\|_{b_\beta,\alpha} > 1$ for all $u \in \mathrm{Lev}_\alpha(p_0(\beta))$, and that $\alpha \in S_\beta$ implies that $\gamma \in b'(\alpha) \subseteq b_\beta(\alpha)$. Consequently, there exists $v \in \mathrm{suc}(u, p_0(\beta))$ with $\gamma \in v(\alpha)$. Note that $v(\alpha) \in [b_\beta(\alpha)]^{<h_\beta(\alpha)}$, and $h_\beta(\alpha) \leq h'(\alpha)$ in virtue of $\alpha \in S_\beta$. It follows that $(p_0(\beta))_v \Vdash$ "$\dot\varphi_\beta(\alpha) = v(\alpha)$", where $\dot\varphi_\beta$ names the generic (b_β, h_β)-slalom. Define $q \leq p_0$ by $q(\zeta) = p_0(\zeta)$ if $\zeta \in \mathcal{A} \setminus \{\beta\}$ and $q(\beta) = (p_0(\beta))_v$, then we see that $q \Vdash$ "$\gamma \in v(\alpha) = \dot\varphi_\beta(\alpha) \wedge \gamma \in b'(\alpha)$", and thus $q \Vdash$ "$\dot f_\xi(\alpha) = \gamma \in \dot\varphi_\beta(\alpha) \cap b'(\alpha) = \dot\varphi'(\alpha)$"

Since α_0 was arbitrary, it follows that $\mathbf{V}[G] \Vdash$ "$f_\xi \in^\infty \varphi'$" for each $\xi < \mu$. □

Lemma 6.2.8

Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of ordinals, with $\mathcal{B}^c = \mathcal{A} \setminus \mathcal{B}$, and let $\langle h_\zeta, b_\zeta \mid \zeta \in \mathcal{A}\rangle$ be a sequence of cofinal increasing cardinal functions such that $h_\zeta \leq^* b_\zeta$ for all $\zeta \in \mathcal{A}$. Let $\overline{\mathbb{Q}} = \prod_{\zeta \in \mathcal{A}}^{\leq\kappa} \mathbb{Q}_\zeta$ and let G be $\overline{\mathbb{Q}}$-generic over $\mathbf{V} \Vdash$ "$2^\kappa = \kappa^+$". If $h' \leq^* b' \in {}^\kappa\kappa$ are cofinal increasing cardinal functions such that $\left(\sup_{\zeta \in \mathcal{B}^c} \left|\mathrm{Loc}_{\leq\alpha}^{b_\zeta,h_\zeta}\right|\right)^{|\alpha|} < h'(\alpha)$ for almost all $\alpha \in \kappa$, then for each $f \in (\prod b')^{\mathbf{V}[G]}$ there exists $\varphi \in (\mathrm{Loc}_\kappa^{b',h'})^{\mathbf{V}[G \restriction \mathcal{B}]}$ and such that $f \in^* \varphi$. ◁

Proof. The proof is essentially that of Lemma 6.1.13. Let us assume that $f \in \mathbf{V}[G] \setminus \mathbf{V}[G \restriction \mathcal{B}]$, since the lemma would be trivially true otherwise. We also assume α is large enough to satisfy the condition on the size of $h'(\alpha)$.

Let $\dot f$ be a $\overline{\mathbb{Q}}$-name for f and let $q \in \overline{\mathbb{Q}}^*$ read $\dot f$ early and assume without loss of generality that q is modest. For the sake of brevity, let us write $\mathcal{Z}^q_{\downarrow\alpha} = \bigcup_{\xi\leq\alpha} \mathcal{Z}^q(\xi)$. Note that:

$$\begin{aligned}
|\mathrm{poss}(q, {\leq}\alpha)| &= \textstyle\prod_{\zeta \in \mathcal{Z}^q_{\downarrow\alpha}} |\mathrm{Lev}_{\alpha+1}(q(\zeta))| \\
&\leq \left(\sup_{\zeta \in \mathcal{Z}^q_{\downarrow\alpha}} (|\mathrm{Lev}_{\alpha+1}(q(\zeta))|)\right)^{|\mathcal{Z}^q_{\downarrow\alpha}|} \\
&\leq \left(\sup_{\zeta \in \mathcal{Z}^q_{\downarrow\alpha}} (|\mathrm{Lev}_{\alpha+1}(q(\zeta))|)\right)^{|\alpha|}.
\end{aligned}$$

Here the last inequality follows from q being modest, which implies that $|\mathcal{Z}^q_{\downarrow\alpha}| \leq \alpha$. Since we wish to construct a name $\dot\varphi$ for some $\varphi \in \mathbf{V}[G \quad \mathcal{B}]$, we see that φ is completely decided by the part of the support in $\mathcal{B}$, and thus we may use the part of the support in $\mathcal{B}^c$ freely to restrict the range of possible values for $f \in \mathbf{V}[G]$ in order to make sure that $q \quad$ "$\dot f \in^* \varphi$", as we did in Lemma 6.1.13. If we restrict our attention to $\mathcal{B}^c$, then we see that

$$\begin{aligned}\left(\sup_{\zeta\in\mathcal{B}^c\cap\mathcal{Z}^q_{\downarrow\alpha}}(|\mathrm{Lev}_{\alpha+1}(q(\zeta))|)\right)^{|\alpha|} &\leq \left(\sup_{\zeta\in\mathcal{B}^c}(|\mathrm{Lev}_{\alpha+1}(q(\zeta))|)\right)^{|\alpha|}\\ &\leq \left(\sup_{\zeta\in\mathcal{B}^c}\left|\mathrm{Loc}^{b_\zeta,h_\zeta}_{\leq\alpha}\right|\right)^{|\alpha|} < h'(\alpha).\end{aligned}$$

This set of possibilities is small enough to define $\varphi \in (\mathrm{Loc}^{b',h'}_\kappa)^{\mathbf{V}[G \ \mathcal{B}]}$. To be precise, we construct a sequence of names $\langle \dot B_\alpha \mid \alpha\in\kappa\rangle$ for sets $B_\alpha \in \mathbf{V}[G \quad \mathcal{B}]$ such that $\mathbf{V}[G \quad \mathcal{B}] \quad$ "$|B_\alpha| < h'(\alpha)$" and such that $q \quad$ "$\dot f(\alpha) \in \dot B_\alpha$".

Since q reads $\dot f$ early, if $\eta \in \mathrm{poss}(q, {\leq}\alpha)$, then let $\gamma_\eta \in \kappa$ be such that $\eta \wedge q \quad$ "$\dot f(\alpha) = \gamma_\eta$".

Given $\eta_{\mathcal{B}} \in \mathrm{poss}(q \quad \mathcal{B}, {\leq}\alpha)$ let $Y(\eta_{\mathcal{B}}) = \{\gamma_\eta \mid \eta \in \mathrm{poss}(q,{\leq}\alpha) \text{ and } \eta \quad \mathcal{B} = \eta_{\mathcal{B}}\}$. Now we define the name $\dot B_\alpha = \{\langle Y(\eta_{\mathcal{B}}), \eta_{\mathcal{B}} \wedge q\rangle \mid \eta_{\mathcal{B}} \in \mathrm{poss}(q \quad \mathcal{B}, {\leq}\alpha)\}$. Since the elements of $Y(\eta_{\mathcal{B}})$ are only dependent on the domain $\mathcal{B}^c$, it follows by the arithmetic from above that $q \quad$ "$|\dot B_\alpha| < h'(\alpha)$", and thus if $\dot\varphi$ names a slalom such that $q \quad$ "$\dot\varphi(\alpha) = \dot B_\alpha$", then $q \quad$ "$\dot f \in^* \dot\varphi \in \mathrm{Loc}^{b',h'}_\kappa$". □

Corollary 6.2.9

Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of ordinals, with $\mathcal{B}^c = \mathcal{A}\setminus\mathcal{B}$, and let $\langle h_\zeta, b_\zeta \mid \zeta\in\mathcal{A}\rangle$ be a sequence of cofinal increasing cardinal functions such that $h_\zeta \leq^* b_\zeta$ for all $\zeta \in \mathcal{A}$. Let $\overline{\mathsf{Q}} = \prod^{\leq\kappa}_{\zeta\in\mathcal{A}} \mathsf{Q}_\zeta$ and let G be $\overline{\mathsf{Q}}$-generic over $\mathbf{V}$. If $h', b', \tilde h, \tilde b \in {}^\kappa\kappa$ are cofinal increasing cardinal functions such that $\left(\sup_{\zeta\in\mathcal{B}^c}\left|\mathrm{Loc}^{b_\zeta,h_\zeta}_{\leq\alpha}\right|\right)^{|\alpha|} < h'(\alpha) \leq \tilde b(\alpha)^{\tilde h(\alpha)} \leq b'(\alpha)$ and $h'(\alpha)\cdot\tilde h(\alpha) < \tilde b(\alpha)$ for almost all $\alpha\in\kappa$, then if $\psi \in (\mathrm{Loc}^{\tilde b,\tilde h}_\kappa)^{\mathbf{V}[G]}$ there exists $g \in (\prod \tilde b)^{\mathbf{V}[G \ \mathcal{B}]}$ such that $g \in^\infty \psi$. ◁

Proof. This follows from Lemma 6.2.8 (as Lemma 6.1.15 follows from Lemma 6.1.13). □

Theorem 6.2.10

There exists a family of parameters $\langle h_\gamma, b_\gamma \mid \gamma\in\kappa\rangle$ such that for any finite $\gamma_n < \cdots < \gamma_0 < \kappa$ and cardinals $\kappa^+ = \lambda_0 < \cdots < \lambda_n$ with $\mathrm{cf}(\lambda_i) > \kappa$ for each $i \in [0,n]$, there exists a forcing extension where $\mathfrak{d}^{b_{\gamma_i},h_{\gamma_i}}_\kappa(\ni^\infty) = \lambda_i$ for each $i\in[0,n]$. ◁

Proof. Let $h_0 \in {}^\kappa\kappa$ be a cofinal increasing cardinal functions such that $h_0(\alpha) \geq |\alpha|$ for all $\alpha\in\kappa$. For each $\gamma\in\kappa$ define h_γ, b_γ recursively as follows: let $h_{\gamma+1}(\alpha) = b_\gamma(\alpha)^{h_\gamma(\alpha)}$ and if γ is limit, let $h_\gamma(\alpha) = \sup_{\xi\in\gamma} h_\xi(\alpha)$, and finally let $b_\gamma(\alpha) = 2^{h_\gamma(\alpha)}$ for all $\gamma\in\kappa$.

Now let $\kappa^+ = \lambda_0 < \lambda_1 < \cdots < \lambda_n$ be a finite sequence of regular cardinals and let $\gamma_0 > \cdots > \gamma_n$ be a decreasing sequence of ordinals. Let $A_1,\dots,A_n$ be disjoint sets of ordinals such that $|A_i| = \lambda_i$ for each $i\in[1,n]$, and let $\mathcal{A} = \bigcup_{i\in[1,n]} A_i$. Now for each $\zeta\in\mathcal{A}$ let $h'_\zeta = h_{\gamma_i}$ and $b'_\zeta = b_{\gamma_i}$ iff $\zeta \in A_i$, and let $\overline{\mathsf{Q}} = \prod^{\leq\kappa}_{\zeta\in\mathcal{A}} \mathsf{Q}_{h'_\zeta,b'_\zeta}$. Let G be $\overline{\mathsf{Q}}$-generic over $\mathbf{V}$ and assume that $\mathbf{V} \quad$ "$2^\kappa = \kappa^+$".

Fix $1 \leq i \leq n$. From Lemma 6.2.8 it follows that $\mathbf{V}[G \quad A_i] \quad$ "$\lambda_i \leq \mathsf{d}_\kappa^{b_{\gamma_i}, h_{\gamma_i}}(\ni^\infty)$". If we let $\mathcal{B} = \bigcup_{i \in [1,i]} A_i$, then also $|\mathcal{B}| = \lambda_i$ and consequently $\mathbf{V}[G \quad \mathcal{B}] \quad$ "$2^\kappa = \lambda_i$". If $i < j \leq n$, then $h_{\gamma_j}, b_{\gamma_j}$ are much smaller than $h_{\gamma_i}, b_{\gamma_i}$, and thus by Corollary 6.2.9 we see that $(\prod b_{\gamma_i})^{\mathbf{V}[G \quad \mathcal{B}]}$ forms a witness to prove that $\mathbf{V}[G] \quad$ "$\mathsf{d}_\kappa^{b_{\gamma_i}, h_{\gamma_i}}(\ni^\infty) \leq \lambda_i$". □

6.3. Open Questions

The main open question is obvious by comparing the classical results to those in this section.

Question 6.3.1

Is it consistent that there exists a family of pairs of functions $\langle (b_\xi, h_\xi) \mid \xi \in \kappa \rangle$ such that the associated antiavoidance numbers $\mathsf{d}_\kappa^{b_\xi, h_\xi}(\ni^\infty)$ are pairwise distinct? If yes, could we prove the same for a family of functions $\langle (b_\xi, h_\xi) \mid \xi \in \kappa^+ \rangle$ or even $\langle (b_\xi, h_\xi) \mid \xi \in 2^\kappa \rangle$? ◁

An essential part of the construction from [KM22] is the ***bigness*** property associated with creature forcing (see e.g. [RS06, Chapter 2]). Bigness is a Ramsey-like property that allows for decreasing the number of successors of some node $u \in T \in \mathsf{Q}_\kappa^{b,h}$, for instance to decide a value of $\dot{f}(\alpha)$ or some name $\dot{f}$, without significantly decreasing the norm $\|\mathrm{suc}(u, T)\|$. In the classical case, one could define a norm on sets in $[b(n)]^{<h(n)}$ that is c-big for certain $c \in \omega$, which means that for any $X \subseteq [b(n)]^{<h(n)}$ and $f : X \to c$ there exists some $X^* \subseteq X$ such that $f \quad X^*$ is constant and $\|X^*\| = \|X\| - \frac{1}{c}$, see for instance [KM22, Lemma 3.10].

Bigness, or at least [KM22, Lemma 3.10], seems hard to generalise to the higher context, since we cannot decrease an infinite norm slightly. After all, decreasing in cardinality can only be done finitely often. On the other hand, allowing a norm to stay constant brings other problems with it. In our definition we required the norm to decrease when successors are removed in order to prove that our forcing is $<\kappa$-closed (Lemma 6.1.5). If one is allowed to remove successors without decreasing the norm, one can partition the set of successors into $\aleph_0$ many parts, resulting in the failure of ω-closure.

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Appendices

A.1. Synopsis of Original Results

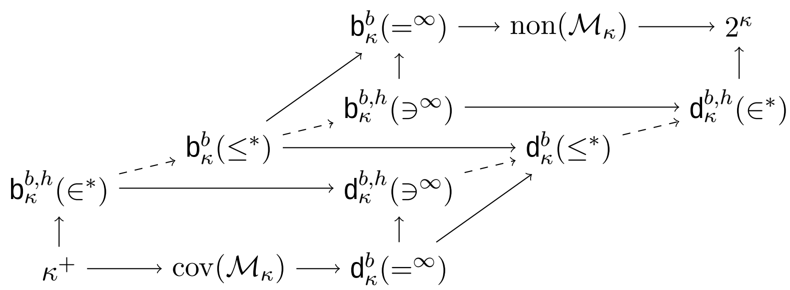
We will present the main results from this dissertation, with a focus on those results found by the author. Naturally, we refer to the results in the dissertation for more information.

Background

Using relational systems (Section 2.3) we may define the ***cardinal characteristics of the Cichoń diagram*** (Definitions 2.4.1 and 2.4.2). By replacing ω (the classical case) by a regular uncountable cardinal κ (the higher case), we define the cardinal characteristics of the ***higher Cichoń diagram*** (Definition 2.4.5). We present a brief overview of what was known prior to the writing of this dissertation about the higher Cichoń diagram in Section 2.5.

Bounded Higher Baire Spaces

By considering a cofinal cardinal function $b \in {}^\kappa\kappa$ for κ inaccessible, we could define a bounded higher Baire space $\prod b = \prod_{\alpha\in\kappa} b(\alpha)$ (Section 3.1), endowed with the $<\kappa$-box topology. We generalise the cardinals of the higher Cichoń diagram by restricting them to $\prod b$ (Definition 3.2.1). We will refer to these as bounded cardinal characteristics. Some such cardinals were previously studied classically, others have no known classical analogue (see Section 3.2). In Section 3.3 we show relations between the bounded cardinal characteristics, summarised in the following diagram. The dashed arrows require $h \leq^* \mathrm{cf}(b)$, and become equalities if also $h =^* b$:



We also show that the unbounded antilocalisation and antiavoidance numbers do not depend on the parameter h.

Corollary 3.3.8
$\mathsf{d}^h_\kappa(\ni^\infty) = \mathrm{cov}(\mathcal{M}_\kappa)$ and $\mathsf{b}^h_\kappa(\ni^\infty) = \mathrm{non}(\mathcal{M}_\kappa)$ for any choice of $h \in {}^\kappa\kappa$. ◁

In Definition 3.3.10 we define cardinal characteristics that are the infima and suprema of sets of cardinal characteristics for every possible bound $b \in {}^\kappa\kappa$. Classically, Rothberger [Rot41] and Miller [Mil81] investigated such infima and suprema. The same strategies used in their work generalise without much problems to the higher Baire space and yields the following two results.

Theorem 3.3.17
$\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathsf{b}_\kappa(\leq^*), \mathsf{inf}_\kappa(=^\infty)\}$ and $\mathrm{cof}(\mathcal{M}_\kappa) = \max\{\mathsf{d}_\kappa(\leq^*), \mathsf{sup}_\kappa(=^\infty)\}$. ◁

Theorem 3.3.19
$\mathsf{b}^h_\kappa(\in^*) = \min\left\{\mathsf{b}_\kappa(\leq^*), \mathsf{inf}^h_\kappa(\in^*)\right\}$ and $\mathsf{d}^h_\kappa(\in^*) = \max\left\{\mathsf{d}_\kappa(\leq^*), \mathsf{sup}^h_\kappa(\in^*)\right\}$ for any $h \in {}^\kappa\kappa$. ◁

We also relate bounded eventually different numbers to the κ-strong measure zero ideal.

Theorem 3.3.16
$\mathrm{non}(\mathcal{SN}_\kappa) \geq \mathsf{inf}_\kappa(=^\infty)$. ◁

We later show that the dual $\mathsf{sup}_\kappa(=^\infty) \geq \mathrm{cov}(\mathcal{SN}_\kappa)$ fails in the κ-Sacks model (Theorem 4.4.9), generalising a classical construction by Goldstern, Judah & Shelah [GJS93]. The same consistency also follows from Chapman & Schürz [CS].

In Section 3.4 we determine the parameters b, h for which bounded cardinal characteristics are trivial, in the sense that they are not consistently strictly between κ^+ and 2^κ. We obtain three trichotomies in Theorems 3.4.2, 3.4.5 and 3.4.16 for $\mathsf{b}^b_\kappa(\leq^*)$, $\mathsf{b}^{b,h}_\kappa(\in^*)$ and $\mathsf{b}^{b,h}_\kappa(\ni^\infty)$ respectively, where in one case these cardinals are $<\kappa$, in one case they are equal to κ and in the remaining case they are consistently strictly between κ^+ and 2^κ.

Partial results for the dual cardinals $\mathsf{d}^b_\kappa(\leq^*)$, $\mathsf{d}^{b,h}_\kappa(\in^*)$ and $\mathsf{d}^{b,h}_\kappa(\ni^\infty)$ are given:

Theorem 3.4.3
If $\lambda < \kappa$ is regular and $\mathrm{cf}(b(\alpha)) = \lambda$ for cofinally many $\alpha \in \kappa$, then $\mathsf{d}^b_\kappa(\leq^*) = 2^\kappa$. ◁

Theorem 3.4.6
If $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$ is cofinal in κ for some $\lambda < \kappa$, then $\mathsf{d}^{b,h}_\kappa(\in^*) = 2^\kappa$. ◁

Theorem 3.4.7
If $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$ is bounded for all $\lambda \in \kappa$ and h is increasing and continuous on a stationary set and $\mathcal{A} \subseteq \prod b$ is an almost disjoint family, then $|\mathcal{A}| \leq \mathsf{d}^{b,h}_\kappa(\in^*)$. ◁

Higher κ-Reals

In Definition 4.2.1 we define a selection of κ-reals (i.e. elements of ${}^\kappa\kappa$) with generic properties over the ground model. The existence of certain types of κ-reals may imply the existence of other types of κ-reals, as per Figure 4.1. We then give an overview in Sections 4.3 and 4.4 of which kinds of κ-reals are added by several well-known $<\kappa$-distributive forcing notions. Most such results should not be attributed to the author, but to other sources or to folklore.

We will highlight that we present forcing notions specifically tailored for the bounded cardinal characteristics to show they are not trivial in the cases mentioned in Section 3.4. Particularly the following three theorems:

Theorem 4.3.19
Let κ be inaccessible and $\mathrm{cf}(b)$ be increasing and discontinuous on a club set, then it is consistent that $\mathsf{b}^b_\kappa(\leq^*) > \kappa^+$ and that $\mathsf{d}^b_\kappa(\leq^*) < 2^\kappa$. ◁

Theorem 4.3.31
Let κ be inaccessible and h be discontinuous on a club set, then it is consistent that $\mathsf{b}_\kappa^{b,h}(\in^*) > \kappa^+$ and that $\mathsf{d}_\kappa^{b,h}(\in^*) < 2^\kappa$. ◁

Theorem 4.3.43
Let κ be inaccessible, let C be a club set such that b is discontinuous on C and let $\mathrm{cf}(b)$ be increasing and discontinuous on $\{\alpha \in C \mid h(\alpha) = b(\alpha)\}$, then it is consistent that $\mathsf{b}_\kappa^{b,h}(\ni^\infty) > \kappa^+$ and that $\mathsf{d}_\kappa^{b,h}(\ni^\infty) < 2^\kappa$. ◁

We also note that, as far as the author is aware, the following theorem regarding κ-eventually different forcing is not found in the literature, although all the tools necessary for the proof are.

Theorem 4.3.39
If κ is weakly compact, then E_κ does not add dominating κ-reals. ◁

Independence Results

Two main results from this thesis regard the consistency of many cardinal characteristics having mutually different values.

In Chapter 5, we introduce an intermediate forcing notion between κ-Sacks and κ-Miller forcing and use it to separate κ many localisation cardinals of the form $\mathsf{d}_\kappa^h(\in^*)$. This answers Question 72 from [BBTFM18].

Corollary 5.2.8
There exists functions h_ξ for each $\xi \in \kappa$ such that for any cardinals $\lambda_\xi > \kappa$ with $\mathrm{cf}(\lambda_\xi) > \kappa$ it is consistent that simultaneously $\mathsf{d}_\kappa^{h_\xi}(\in^*) = \lambda_\xi$ for all $\xi \in \kappa$. ◁

The proof relies on a family of κ many disjoint stationary sets. In Section 5.3 we improve this result further using an almost disjoint family of stationary sets to obtain:

Theorem 5.3.9
Assuming $2^\kappa = \kappa^+$ and $_\kappa$, there exists a family of functions $\{g_\eta \mid \eta \in \kappa^+\} \subseteq {}^\kappa\kappa$ such that for any increasing sequence $\langle \lambda_\xi \mid \xi \in \kappa^+ \rangle$ of cardinals with $\kappa < \mathrm{cf}(\lambda_\xi)$ and any function $\sigma : \kappa^+ \to \kappa^+$, there exists a forcing extension in which $\mathsf{d}_\kappa^{g_\eta}(\in^*) = \lambda_{\sigma(\eta)}$ for all $\eta \in \kappa^+$. ◁

In Chapter 6 we attempt to prove a similar result for antiavoidance cardinals $\mathsf{d}_\kappa^{b,h}(\ni^\infty)$ by generalising a forcing construction described in [KM22]. Although we do not separate infinitely many cardinals, we can give a set of cardinal characteristics of size κ such that any finite subset can be forced to become pairwise distinct.

Theorem 6.2.10
There exists a family of parameters $\langle h_\gamma, b_\gamma \mid \gamma \in \kappa \rangle$ such that for any finite $\gamma_n < \dots < \gamma_0 < \kappa$ and cardinals $\kappa^+ = \lambda_0 < \dots < \lambda_n$ with $\mathrm{cf}(\lambda_i) > \kappa$ for each $i \in [0, n]$, there exists a forcing extension where $\mathsf{d}_\kappa^{b_{\gamma_i}, h_{\gamma_i}}(\ni^\infty) = \lambda_i$ for each $i \in [0, n]$. ◁

A.2. Zusammenfassung der Ergebnisse

This section is a translation of Appendix A.1. The translation was made with assistance of translation software, manual corrections were made by the author.

Wir präsentieren die Hauptergebnisse dieser Dissertation, insbesondere die vom Autor bewiesenen Ergebnisse. Weitere Details finden sich in der Dissertation.

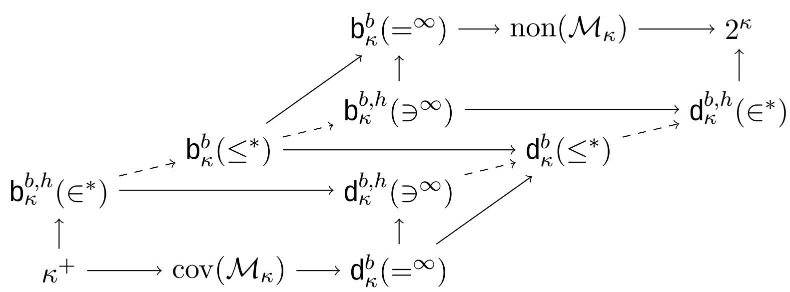
Hintergrund

Mit Hilfe von relationalen Systemen (***relational system***, Abschnitt 2.3) können wir die ***Kardinalzahlinvarianten des Cichoń-Diagramms*** (Definitionen 2.4.1 und 2.4.2) definieren. Indem wir ω (den klassischen Fall) durch eine überabzählbare, reguläre Kardinal κ (den höheren Fall) ersetzen, definieren wir die Kardinalzahlinvarianten des ***höheren Cichoń-Diagramms*** (Definition 2.4.5). Wir geben einen kurzen Überblick darüber, was vor dem Verfassen dieser Dissertation über das höhere Cichoń-Diagramm bekannt war, in Abschnitt 2.5.

Beschränkte Höhere Bairesche Räume

Indem wir eine konfinale Funktion $b \in {}^\kappa\kappa$ mit Kardinalzahlen als Werten für ein unerreichbares κ betrachten, könnten wir einen beschränkten höheren Baireschen Raum $\prod b = \prod_{\alpha\in\kappa} b(\alpha)$ definieren (Abschnitt 3.1), den wir mit der $<\kappa$-Boxtopologie ausstatten. Wir verallgemeinern die Kardinalzahlinvarianten des höheren Cichoń-Diagramms, indem wir sie auf $\prod b$ beschränken (Definition 3.2.1). Wir werden diese als beschränkte Kardinalzahlinvarianten bezeichnen.

Einige dieser Kardinalzahlinvarianten wurden zuvor klassisch untersucht. Andere haben kein bekanntes klassisches Analogon (siehe Abschnitt 3.2). In Abschnitt 3.3 zeigen wir Beziehungen zwischen den beschränkten Kardinalzahlinvarianten, zusammengefasst im folgenden Diagramm. Die gestrichelten Pfeile benötigen $h \leq^* \mathrm{cf}(b)$ und werden zu Gleichungen, wenn auch $h =^* b$:



Wir zeigen auch, dass die unbeschränkte Antilokalisierungszahl und die unbeschränkte Antivermeidungszahl nicht vom Parameter h abhängen:

Korollar 3.3.8
$\mathfrak{d}^h_\kappa(\ni^\infty) = \mathrm{cov}(\mathcal{M}_\kappa)$ und $\mathfrak{b}^h_\kappa(\ni^\infty) = \mathrm{non}(\mathcal{M}_\kappa)$ für jede Wahl von $h \in {}^\kappa\kappa$. ◁

In Definition 3.3.10 definieren wir Kardinalzahlinvarianten, welche die Infima und Suprema von Mengen von beschränkten Kardinalzahlinvarianten für jede mögliche Schranke $b \in {}^\kappa\kappa$ sind.

Im klassischen Falle untersuchten Rothberger [Rot41] und Miller [Mil81] solche Infima und Suprema. Die gleichen Strategien, die in ihrer Arbeit verwendet wurden, kann man ohne größere Schwierigkeiten auf den höheren Baireschen Raum verallgemeinern und führen zu den folgenden zwei Ergebnissen.

Satz 3.3.17
$\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathsf{b}_\kappa(\leq^*), \mathsf{inf}\kappa(=^\infty)\}$ und $\mathrm{cof}(\mathcal{M}\kappa) = \max\{\mathsf{d}_\kappa(\leq^*), \mathsf{sup}_\kappa(=^\infty)\}$. ◁

Satz 3.3.19
$\mathsf{b}^h_\kappa(\in^*) = \min\left\{\mathsf{b}_\kappa(\leq^*), \mathsf{inf}\kappa^h(\in^*)\right\}$ und $\mathsf{d}^h_\kappa(\in^*) = \max\left\{\mathsf{d}_\kappa(\leq^*), \mathsf{sup}\kappa^h(\in^*)\right\}$ für jedes $h \in {}^\kappa\kappa$. ◁

Wir stellen auch eine Beziehung zwischen den beschränkten Kardinalzahlinvarianten in Bezug auf ingerndwann-Unterschiedlichkeit (*eventual di erence*, $=^\infty$) und dem Ideal der κ-starken Nullmengen her.

Satz 3.3.16
$\mathrm{non}(\mathcal{SN}_\kappa) \geq \mathsf{inf}_\kappa(=^\infty)$. ◁

Wir zeigen auch, dass die duale Aussage $\mathsf{sup}_\kappa(=^\infty) \geq \mathrm{cov}(\mathcal{SN}_\kappa)$ im κ-Sacks-Modell scheitert (Satz 4.4.9), eine Verallgemeinerung einer klassischen Konstruktion von Goldstern, Judah & Shelah [GJS93]. Die gleiche Konsistenz folgt auch aus Chapman & Schürz [CS].

In Abschnitt 3.4 bestimmen wir die Parameter b, h, für die die beschränkten Kardinalzahlinvarianten trivial sind, im Sinne, dass sie nicht konsistent strikt zwischen κ^+ und 2^κ liegen. Wir erhalten drei Trichotomien in Sätze 3.4.2, 3.4.5 und 3.4.16 für $\mathsf{b}^b_\kappa(\leq^*)$, $\mathsf{b}^{b,h}_\kappa(\in^*)$ und $\mathsf{b}^{b,h}_\kappa(\ni^\infty)$, wobei in einem Fall diese Kardinalzahlen $<\kappa$ sind, in einem anderen Fall gleich κ und im verbleibenden Fall konsistent strikt zwischen κ^+ und 2^κ liegen.

Wir erhalten Teilergebnisse für die dualen Kardinalzahlen $\mathsf{d}^b_\kappa(\leq^*)$, $\mathsf{d}^{b,h}_\kappa(\in^*)$ und $\mathsf{d}^{b,h}_\kappa(\ni^\infty)$:

Satz 3.4.3
Wenn $\lambda < \kappa$ regulär ist und $\mathrm{cf}(b(\alpha)) = \lambda$ für konfinal viele $\alpha \in \kappa$, dann ist $\mathsf{d}^b_\kappa(\leq^*) = 2^\kappa$. ◁

Satz 3.4.6
Wenn $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$ in κ konfinal ist für ein $\lambda < \kappa$, dann ist $\mathsf{d}^{b,h}_\kappa(\in^*) = 2^\kappa$. ◁

Satz 3.4.7
Wenn $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$ für alle $\lambda \in \kappa$ beschränkt ist und h auf einer stationären Menge monoton steigend und stetig ist und $\mathcal{A} \subseteq \prod b$ eine fast disjunkte Familie ist, dann ist $|\mathcal{A}| \leq \mathsf{d}^{b,h}_\kappa(\in^*)$. ◁

Höhere κ-Reelle Zahlen

In Definition 4.2.1 definieren wir einige κ-reellen Zahlen (d.h. Elemente von ${}^\kappa\kappa$) mit generischen Eigenschaften über dem Grundmodell. Die Existenz bestimmter Typen von κ-reellen Zahlen kann die Existenz anderer Typen von κ-reellen Zahlen implizieren, wie in Abbildung 4.1 dargestellt ist. Anschließend geben wir in den Abschnitte 4.3 und 4.4 einen Überblick darüber, welche Arten von κ-reellen Zahlen in der generischen Erweiterung von mehreren bekannten $<\kappa$-distributiven partiellen Ordnungen vorkommen. Die meisten solcher Ergebnisse sollten nicht dem Autor zugeschrieben werden, sondern anderen Quellen oder sind sogenannte Folklore-Resultate.

Wir betonen, dass wir für die beschränkten Kardinalzahlinvarianten spezifische partielle Ordnungen konstruieren, um zu zeigen, dass sie Abschnitt 3.4 genannten Fällen nicht trivial sind; insbesondere beweisen wir die folgenden drei Sätze:

Satz 4.3.19
Sei κ unerreichbar und $\mathrm{cf}(b)$ steigend und unstetig auf einer club-Menge, dann ist es konsistent, dass $\mathfrak{b}^b_\kappa(\leq^*) > \kappa^+$ und dass $\mathfrak{d}^b_\kappa(\leq^*) < 2^\kappa$ ist. ◁

Satz 4.3.31
Sei κ unerreichbar und h unstetig auf einer club-Menge, dann ist es konsistent, dass $\mathfrak{b}^{b,h}_\kappa(\in^*) > \kappa^+$ und dass $\mathfrak{d}^{b,h}_\kappa(\in^*) < 2^\kappa$ ist. ◁

Satz 4.3.43
Sei κ unerreichbar, sei C eine club-Menge, auf der b unstetig ist, und sei $\mathrm{cf}(b)$ steigend und unstetig auf $\{\alpha \in C \mid h(\alpha) = b(\alpha)\}$, dann ist es konsistent, dass $\mathfrak{b}^{b,h}_\kappa(\ni^\infty) > \kappa^+$ und dass $\mathfrak{d}^{b,h}_\kappa(\ni^\infty) < 2^\kappa$ ist. ◁

Der folgende Satz über die κ-Irgendwann-Unterschiedlich-Ordnung (κ-*eventually di erent forcing*) folgt aus bekannten Ergebnissen, ist aber nach Kenntnis des Autors nicht in der veröffentlichen Literatur zu finden.

Satz 4.3.39
Wenn κ schwach kompakt ist, dann fügt $\mathbb{E}_\kappa$ keine dominierenden κ-reellen Zahlen hinzu. ◁

Unabhängigkeitsergebnisse

Zwei Hauptergebnisse dieser Dissertation betreffen die Konsistenz der Aussage, dass viele Kardinalzahlinvarianten gegenseitig unterschiedliche Werte haben.

In Kapitel 5 führen wir eine partielle Ordnung ein, die zwischen κ-Sacks- und κ-Miller-Forcing liegt und verwenden sie, um κ viele Lokalisationskardinalzahlinvarianten der Form $\mathfrak{d}^h_\kappa(\in^*)$ zu trennen. Dies beantwortet Frage 72 aus [BBTFM18].

Korollar 5.2.8
Es existieren Funktionen h_ξ für jedes $\xi \in \kappa$, so dass es für beliebige Kardinalzahlen $\lambda_\xi > \kappa$ mit $\mathrm{cf}(\lambda_\xi) > \kappa$ konsistent ist, dass gleichzeitig $\mathfrak{d}^{h_\xi}_\kappa(\in^*) = \lambda_\xi$ für alle $\xi \in \kappa$ gilt. ◁

Der Beweis verwendet eine Familie von κ vielen disjunkten stationären Mengen. In Abschnitt 5.3 verbessern wir dieses Ergebnis unter Verwendung einer fast disjunkten Familie von stationären Mengen und erhalten:

Satz 5.3.9
Angenommen, es gilt $2^\kappa = \kappa^+$ und ${}_\kappa$, dann existiert eine Familie $\{g_\eta \mid \eta \in \kappa^+\} \subseteq {}^\kappa\kappa$, so dass

für jede monoton steigende Folge $\langle \lambda_\xi \mid \xi \in \kappa^+ \rangle$ von Kardinalzahlen mit $\kappa < \mathrm{cf}(\lambda_\xi)$ und jede Funktion $\sigma : \kappa^+ \to \kappa^+$, es eine generische Erweiterung gibt, in der $\mathsf{d}_\kappa^{g_\eta}(\in^*) = \lambda_{\sigma(\eta)}$ für alle $\eta \in \kappa^+$ gilt. $\triangleleft$

In Kapitel 6 versuchen wir, ein ähnliches Ergebnis für die Antivermeidungskardinalzahlinvarianten $\mathsf{d}_\kappa^{b,h}(\ni^\infty)$ zu beweisen, indem wir eine Forcing-Konstruktion verallgemeinern, die in [KM22] beschrieben wird. Obwohl wir nicht unendlich viele Kardinalzahlen trennen, können wir eine Menge der Größe κ von Kardinalzahlinvarianten angeben, so dass jede endliche Teilmenge erzwungen werden kann, paarweise unterschiedlich zu werden.

Satz 6.2.10
Es existiert eine Familie von Parametern $\langle h_\gamma, b_\gamma \mid \gamma \in \kappa \rangle$, so dass für beliebige $n \in \omega$, Ordinalzahlen $\gamma_n < \cdots < \gamma_0 < \kappa$ und Kardinalzahlen $\kappa^+ = \lambda_0 < \cdots < \lambda_n$ mit $\mathrm{cf}(\lambda_i) > \kappa$ für jedes $i \in [0, n]$, es eine Forcing-Erweiterung gibt, in der $\mathsf{d}_\kappa^{b_{\gamma_i}, h_{\gamma_i}}(\ni^\infty) = \lambda_i$ für jedes $i \in [0, n]$ gilt. $\triangleleft$